INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING
Int. J. Numer. Meth. Engng 2009; **79**:253–289
Published online 27 February 2009 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/nme.2576

REVIEW

Group-theoretic exploitations of symmetry in computational solid and structural mechanics

A. Zingoni*, †

Department of Civil Engineering, University of Cape Town, Rondebosch 7701, Cape Town, South Africa

SUMMARY

The use of group theory in simplifying the study of problems involving symmetry is a well-established approach in various branches of physics and chemistry, and major applications in these areas date back more than 70 years. Within the engineering disciplines, the search for more systematic and more efficient strategies for exploiting symmetry in the computational problems of solid and structural mechanics has led to the development of group-theoretic methods over the past 40 years. This paper reviews the advances made in the application of group theory in areas such as bifurcation analysis, vibration analysis and finite element analysis, and summarizes the various implementation procedures currently available. Illustrative examples of typical solution procedures are drawn from recent work of the author. It is shown how the group-theoretic approach, through the characteristic vector-space decomposition, enables considerable simplifications and reductions in computational effort to be achieved. In many cases, group-theoretic considerations also allow valuable insights on the behaviour or properties of a system to be gained, before any actual calculations are carried out. Copyright © 2009 John Wiley & Sons, Ltd.

Received 6 August 2008; Accepted 16 January 2009

KEY WORDS: group theory; symmetry group; representation theory; solid mechanics; structural mechanics; bifurcation analysis; vibration analysis; finite element formulation; structures; symmetry; vibrations; finite element methods

1. INTRODUCTION

Nature exhibits symmetry on a level ranging from the microscopic to the macroscopic, from atoms, molecules and crystals, to planetary systems and cosmic structures. Considerations of symmetry have played a major role in shaping the man-made world of architecture and engineering construction, and in the design of all sorts of machinery. In simple terms, a body or a structure is said to exhibit symmetry if one part of it is similar or identical to another part relative to a centre, an axis or a plane. More formally, a body or a structure is said to exhibit symmetry if it

*Correspondence to: A. Zingoni, Department of Civil Engineering, University of Cape Town, Rondebosch 7701, Cape Town, South Africa.
†E-mail: alphose.zingoni@uct.ac.za

can be turned into one or more new configurations physically indistinguishable from the initial configuration through the application of one or more symmetry operations such as reflections in planes, rotations about axes, or inversions about some centre.

Although wider in its scope, group theory has been described as 'the mathematics of symmetry' [1], because it is in the systematic study of symmetry and its consequences that this branch of mathematics has found its most fruitful applications. Of particular interest here are symmetry groups (that is, groups whose elements are symmetry operations). Applications of group theory to problems in physics and chemistry are well known. They have been demonstrated in areas such as quantum mechanics, molecular chemistry and crystallography [2–8].

Up to about 1970, and as noted by Glockner [1], only isolated cases existed in which researchers had exploited group-theoretic methods in structural mechanics, unlike in the domains of physics and chemistry, where group-theoretic techniques as a tool for handling symmetry were already well established. Considerable progress has been made since that time, and as will be seen in this paper, fruitful applications have arisen in areas such as vibration, stability, bifurcation and finite element analysis, among others.

Traditionally, structural analysis has exploited symmetry via two well-known approaches. The first makes use of the concepts of 'symmetry' and 'antisymmetry', and is applicable to structures that exhibit planes or axes of symmetry. Simplification in the analysis is achieved by dividing the structure into substructures through the axes or planes of symmetry, and then analysing only one substructure. Appropriate boundary conditions have to be imposed along the planes of division, to simulate the continuity of the full structure.

Instead of partitioning the physical system into substructures through symmetry planes, one can alternatively manipulate the stiffness matrix of the complete system and partition it, enabling the analysis of the entire structure to be effected on the basis of a much smaller matrix. Clearly, this implies reduced computer storage requirements and reduced computational time. If s represents the number of symmetry axes (planes) of a symmetric skeletal structure, then according to Kardestuncer and Berg [9], the stiffness matrix of such a structure will possess $(s+1)$ symmetry axes, and only 0.25^s of the stiffness matrix will be needed to perform the analysis for the entire structure.

The second approach makes use of the concept of 'cyclic' or 'rotational' symmetry. Here the structure consists of identical substructures forming a rotationally periodic pattern (that is, the substructure is repeated a number of times as one sweeps through a full circle around the axis of rotational symmetry of the structure). For structures with this type of symmetry, discrete Fourier methods can be applied to simplify the analysis [10–12].

Clearly, these methods cannot take into account the types of symmetry for which they are not suited, and in the case of structural configurations possessing complex or mixed symmetry properties, only a portion of the total symmetry can be exploited. As we shall see, the group-theoretic approach, based on the theory of symmetry groups and associated representation theory, has the capability of taking into account *all* the symmetry properties of a configuration in a systematic manner. This has pointed to the possibility of dramatic reductions in computational effort, particularly in the case of large-scale problems that are rich in symmetry. Furthermore, in those cases where engineering structures exhibit small deviations from perfect symmetry, group theory can still be employed to simplify the analysis [13].

Before considering specific applications of group theory in the various areas of solid and structural mechanics, let us begin by summarizing the basic concepts of symmetry groups and their representation. Much of the behaviour of a symmetric physical system can be explained in terms of the intrinsic properties of the associated symmetry group and its representations. It is for

DOI: 10.1002/nme

this reason that the most important aspects of the theory are included here. For a more detailed treatment of group theory and symmetry groups, the reader is referred to specialist texts on the subject (for example, Reference [4]).

2. BASIC CONCEPTS OF SYMMETRY GROUPS AND REPRESENTATION THEORY

A set of elements $\{a, b, c, \ldots, g, \ldots\}$ comprises a group G with respect to a binary operation (for example, multiplication) if the following axioms are satisfied [4]:

(i) The product c of any two elements a and b of the group, denoted by $c = ab$, must be a unique element which also belongs to the group.
(ii) Among the elements of G, there must exist an identity element e which, when multiplied with any element a of the group, leaves the element unchanged: $ea = ae = a$.
(iii) For every element a of G, there must exist another element d also belonging to the group G, such that $ad = da = e$; d is referred to as the inverse of a, and denoted by a^{-1}.
(iv) The order of the multiplication of three or more elements of G does not affect the result (that is, multiplication is associative): $a(bc) = (ab)c$.

Groups for which $ab = ba$ (for any elements a and b of the group) are called Abelian groups. When all elements of G are symmetry operations, then the group G is called a *symmetry group*. Symmetry operations are transformations which bring an object into coincidence with itself, and leaves it indistinguishable from its original configuration. For finite objects (whose symmetry groups are referred to as *point groups*), symmetry operations are of the following types:

(i) reflections in planes of symmetry, which we will denote by σ_l, where l is the plane of symmetry,
(ii) rotations about an axis of symmetry, which we will denote by c_n, if the angle of rotation is $2\pi/n$,
(iii) rotation–reflections, which we will denote by s_n; these represent a rotation through an angle $2\pi/n$, combined with a reflection in the plane perpendicular to the axis of rotation. The special case of s_n when $n = 2$ is sometimes referred to as an 'inversion' (symbol i). It represents a reflection through the *centre of symmetry* (that is, the one point of a finite object which remains unmoved by all symmetry operations).

The present review will concern symmetry groups associated with common structural configurations, which may or may not be Abelian groups. More detailed classification of symmetry groups is usually based on the types of symmetry elements making them up. For example, groups denoted by C_n and C_{nv} all possess a single n-fold axis of rotational symmetry (giving n rotation elements, one of which is equivalent to the identity element e), with the C_{nv} groups possessing an additional n reflection elements. Groups C_n and C_{nv} are of order n and $2n$, respectively, the order of a group being simply the total number of elements comprising it. Groups C_n are referred to as *cyclic* groups. Figure 1 illustrates C_n and C_{nv} configurations for $n = 8$.

An element a of a group G is said to be conjugate to the element b belonging to the same group if there exists an element g in G, such that $a = g^{-1}bg$. The collection of all elements formed by evaluating $g^{-1}bg$ for all g in the group is called the *class* of b.

Let us next turn our attention to the representation of symmetry groups. If a set of symmetry operators $\{a, b, c, \ldots, g, \ldots\}$ in an n-dimensional vector space V constitutes a group G in the

DOI: 10.1002/nme

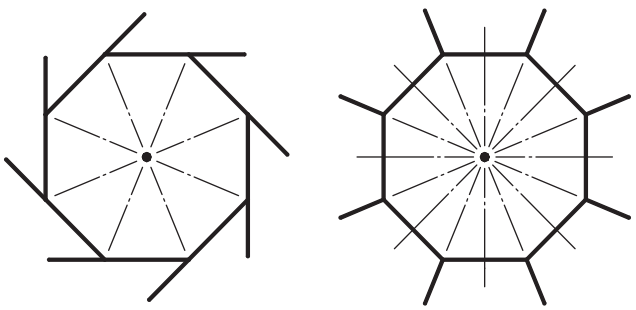

Figure 1. (a) C_8 configuration and (b) C_{8v} configuration.

sense already defined, then the set $R(G)$ of matrices describing all the symmetry operators of G, with respect to a particular basis of the n-dimensional vector space, constitutes an n-dimensional *representation* of G. The *trace* of a matrix (i.e. the sum of the diagonal elements) representing an operator a is called the *character* of a, denoted by $\chi(a)$.

If the basis of V is changed, a new set of matrices $R'(G)$, also constituting a representation of the group G, results. Since a basis transformation does not change the trace of a matrix representing a linear operator in an n-dimensional vector space, this second representation of G will have the same set of characters as the first representation.

Suppose now that a basis is found with respect to which the matrices of the group representation are expressed as direct sums of submatrices that no further change of basis can reduce to matrices of smaller dimensions

$$R(G)=\left[\begin{array}{c|c|c|c} R^{(1)}G & 0 & \cdot & 0 \\ \hline 0 & R^{(2)}G & \cdot & 0 \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline 0 & 0 & \cdot & R^{(k)}G \end{array}\right] \tag{1}$$

Then sets $R^{(\mu)}(G)$ ($\mu=1, 2, \ldots, k$) of these submatrices (a set covering all the symmetry operators of the group G) constitute *irreducible* representations. The representation $R(G)$ is said to be fully reducible, and Equation (1) may be rewritten as

$$R(G)=R^{(1)}(G)+R^{(2)}(G)+\cdots+R^{(k)}(G) \tag{2}$$

In the above equations, $R(G)$ is a matrix of size $n\times n$, whereas $R^{(1)}(G), R^{(2)}(G), \ldots, R^{(k)}(G)$ are the block-diagonal submatrices of $R(G)$. These blocks are of size $n_1\times n_1, n_2\times n_2, \ldots, n_k\times n_k$, respectively, where $n_1+n_2+\cdots+n_k=n$.

The above process of reduction (culminating in irreducible representations) effectively divides the vector space V of the problem into a number of *group-invariant subspaces* $U^{(i)}$, such that none of these subspaces can be divided into further group-invariant subspaces of smaller dimensions. According to representation theory, if a group G of order m has k classes, then the number of different irreducible representations is finite and equal to k.

DOI: 10.1002/nme

Table I. General format for character tables.

G	K_1	K_2	·	K_j	·	K_k
$R^{(1)}$	$\chi_1^{(1)}$	$\chi_2^{(1)}$	·	$\chi_j^{(1)}$	·	$\chi_k^{(1)}$
$R^{(2)}$	$\chi_1^{(2)}$	$\chi_2^{(2)}$	·	$\chi_j^{(2)}$	·	$\chi_k^{(2)}$
·	·	·	·	·	·	·
$R^{(i)}$	$\chi_1^{(i)}$	$\chi_2^{(i)}$	·	$\chi_j^{(i)}$	·	$\chi_k^{(i)}$
·	·	·	·	·	·	·
$R^{(k)}$	$\chi_1^{(k)}$	$\chi_2^{(k)}$	·	$\chi_j^{(k)}$	·	$\chi_k^{(k)}$

In any irreducible representation, all group elements belonging to the same class have the same character, since traces of conjugate elements are equal. Denoting the k different classes of a group G by $K_1, K_2, \ldots, K_k$, the k different irreducible representations by $R^{(1)}, R^{(2)}, \ldots, R^{(k)}$, and the character which belongs to the class K_j in the representation $R^{(i)}$ by $\chi_j^{(i)}$, characters can be listed in the format depicted in Table I. Comprehensive lists of character tables for point groups are widely available [4, 5]. The sets of characters $\chi_j^{(i)}$ $(j=1,2,\ldots,k)$ for different irreducible representations are always different (i.e. no two sets are the same).

If we introduce addition as a second operation into a group G, we can create the *group algebra* Z. The group algebra, whose elements are all the linear combinations of group elements, has the properties of a vector space. In the group algebra Z, the elements of G constitute a basis of Z, since they are linearly independent. All elements of G that commute with every element of Z comprise the *centre of the group algebra*. Class sums (i.e. sums of group elements belonging to the same class) form a basis of the centre of the group algebra, and the dimension of the centre equals the number of classes.

A crucial concept in the application of group theory is that of an *idempotent.* Idempotents $P^{(i)}$ of the group algebra are its non-zero elements which satisfy the relation $P^{(i)}P^{(i)}=P^{(i)}$. Orthogonal idempotents satisfy the relation $P^{(i)}P^{(j)}=0$ if $i\neq j$. They are linearly independent; the sum of orthogonal idempotents is also an idempotent. Idempotents of the centre of the group algebra are linear combinations of class sums. An idempotent $P^{(i)}$ corresponding to the irreducible representation $R^{(i)}$, by operating on vectors of the space V, nullifies every vector which does not belong to the subspace $U^{(i)}$ of $R^{(i)}$. Thus, out of all the vectors belonging to the group-invariant subspaces $U^{(1)}, U^{(2)}, \ldots, U^{(k)}$, the operator $P^{(i)}$ selects all vectors belonging to the subspace $U^{(i)}$ (which all have a definite symmetry-type characteristic of the subspace), and therefore acts as a *projection operator* [4] of the subspace $U^{(i)}$. For instance, when applied as operators upon the positional functions $f_1, f_2, \ldots, f_n$ of a problem, idempotents generate the *symmetry-adapted positional functions* for their respective subspaces, enabling basis vectors for the various subspaces to be written down.

The orthogonal idempotents of the centre of the group algebra ($P^{(i)}$ for the subspace $U^{(i)}$, for $i=1,2,\ldots,k$) can be written down directly from the character table of the group, using the relation [14]

$$P^{(i)}=\frac{\delta^{(i)}}{m}\sum_g \chi^{(i)}(g^{-1})g \tag{3}$$

where m is the order of G (i.e. the number of elements of G), $\delta^{(i)}$ is the dimension of the ith irreducible representation, given by $\delta^{(i)}=\chi^{(i)}(e)$, the first character of the ith row of the character

DOI: 10.1002/nme

table), $\chi^{(i)}$ is a character of the ith irreducible representation, g is an element of G, and g^{-1} its inverse.

Summarizing, if a physical system possesses symmetry properties which can be described by a group, decomposition of the vector space V of the system on the basis of the above procedure results in a block-diagonal form of the matrix of equations describing the behaviour of the system, allowing separation into independent sets of equations each corresponding to a particular subspace of V.

Various classes of problems in structural mechanics will be considered in the following sections. While group-theoretic formulations will differ from one application to another, the basic ideas will remain essentially the same.

In this paper, we are not concerned with applications of group theory to the solution of differential equations as an end in itself, but rather with applications of group theory to the solution of physical problems exhibiting symmetry (which may or may not involve differential equations). However, since differential equations feature so frequently in solid and structural mechanics, we will just briefly mention here their connection with group theory. Applications of group theory to the solution of differential equations had their origins about 125 years ago, in the work of Sophus Lie. Many papers have since been written on the subject. For a comprehensive treatment of the subject, the reader is referred to References [15, 16].

A symmetry of a differential equation is a transformation which maps any solution of the differential equation to another solution. A Lie symmetry is characterized by an infinitesimal transformation which leaves a differential equation invariant under the transformation of all independent and dependent variables. The Lie symmetries of differential equations form a group, in the sense defined earlier. Applications of Lie groups in the theory of differential equations include: (i) mapping solutions of differential equations to other solutions; (ii) reduction of the order of ordinary differential equations and of the number of independent variables of partial differential equations; (iii) construction of invariant solutions; and (iv) construction of conservation laws. Problems of structural mechanics recently studied on the basis of Lie groups include the Timoshenko beam equations [17], the theory of shells and rods [18], the theory of affine rods [19], and the von Karman equations for large deflections of thin elastic plates [20].

3. BIFURCATION PROBLEMS

Renton's paper of 1964, on the buckling of symmetrical frameworks [21], ranks among the earliest applications of group theory to problems of structural stability. Since then bifurcation theory has emerged as one of the most fruitful areas of application of group-theoretic methods in structural mechanics. A group-theoretic bifurcation theory has been given by Sattinger [22], and detailed considerations of the singularity behaviour of symmetric configurations, and how group-theoretic bifurcation theory accounts for this, may be seen in the work of Golubitsky and Schaeffer [23, 24]. The reader interested in a comprehensive treatment of groups and bifurcation theory is referred to these books. However, the main concern of this work was problems of local (as opposed to global) bifurcation.

With regard to specific problems, the post-buckling secondary-bifurcation behaviour of an axially compressed bar of constant rectangular cross section has been investigated by Buzano [25], who employed group-theoretic concepts to simplify the analysis and generate the required bifurcation diagrams. The symmetry group C_{2v} (describing the rectangular shape of the cross section) forms the basis for the formulation of the problem. The author adopts the ratio of the sides of the

DOI: 10.1002/nme

rectangular section as a parameter, and when this ratio is made to equal one, the results for a square cross section are directly deduced without having to re-formulate the problem in terms of the C_{4v} symmetry of the square.

The development of group-theoretic procedures for *global* (as opposed to *local*) bifurcation analysis has been the subject of many fairly recent investigations. It is recognized that symmetric structures have a much more complex bifurcation behaviour than non-symmetric structures. Multiple critical points, where two or more eigenvalues vanish simultaneously, are inherent in symmetric structures [26]. For structural configurations of this kind, the group-theoretic approach is naturally suited to untangling the problem and simplifying the analysis. Typically, the procedure allows the tangent-stiffness matrix of the structure to be brought into block-diagonal form, facilitating tests for singularity of the tangent-stiffness matrix [27], or eliminating problems of numerical ill-conditioning usually associated with the bifurcation analysis of thin shells [28]. These studies bring out three major attributes of group-theoretic procedures: (i) computational simplification (such as block-diagonalization of the stiffness matrix), (ii) circumvention of computational problems (such as numerical ill-conditioning), and (iii) insights on structural behaviour (e.g. prediction of bifurcation patterns).

Lattice domes and related space structures have been the subject of several recent group-theoretic investigations [26, 27, 29], owing to the high degree of symmetry they naturally possess. Of particular interest have been structural configurations belonging to the dihedral groups D_n (i.e. groups describing the symmetry of regular polygons), which are very common. Elements of a group D_n are as follows:

$$D_n = \{e, c_n, \ldots, c_n^{n-1}, \sigma, \sigma c_n, \ldots, \sigma c_n^{n-1}\} \tag{4}$$

where c_n^j $(j = 1, \ldots, n-1)$ represents a clockwise rotation about the z axis through an angle of $2\pi j/n$, σ is a reflection relative to the xz plane and $\sigma^2 = c_n^n = e$ (identity). Here, a configuration is assumed to lie in the 3-dimensional Cartesian coordinate system, with the z axis being vertical and the xy plane being horizontal and containing the plan projection of the configuration. Figure 2 shows space-truss domes with D_3 and D_6 symmetries.

It is clear that bifurcation patterns of D_n configurations will generally be less symmetric than the configurations themselves (bifurcation being a process of 'symmetry breaking'). The patterns will in general retain only part of the original symmetry, represented by the subgroups of D_n (which are lower-order dihedral groups) and cyclic groups C_m (whose order m is an integer factor of n). For instance, the subgroups of the group D_3 are:

$$D_3, \quad C_3 = \{e, c_3, c_3^2\}, \quad D_1^j = \{e, \sigma c_3^{j-1}\} \ (j = 1, 2, 3), \quad C_1 = \{e\}$$

Healey [29] has applied group invariance to the global bifurcation problem of symmetric structures, illustrating the procedure by the example of a lattice dome with hexagonal symmetry (Figure 2(b)). The basic idea is the formulation of a reduced problem having fewer unknowns than the original problem; solutions of the reduced problem are exact solutions of the full problem. The paper demonstrates that a reduced problem does not possess a singularity at a (symmetry-breaking) bifurcation point of the full problem, circumventing the usual numerical problems near bifurcation points. The study achieves a considerable reduction in computational effort in the determination of several global solution branches, enabling the accurate computation of a fairly large number of singular points. Although only one particular example is considered, this study is an excellent demonstration of the power of the group-theoretic approach in not only reducing the computations

DOI: 10.1002/nme

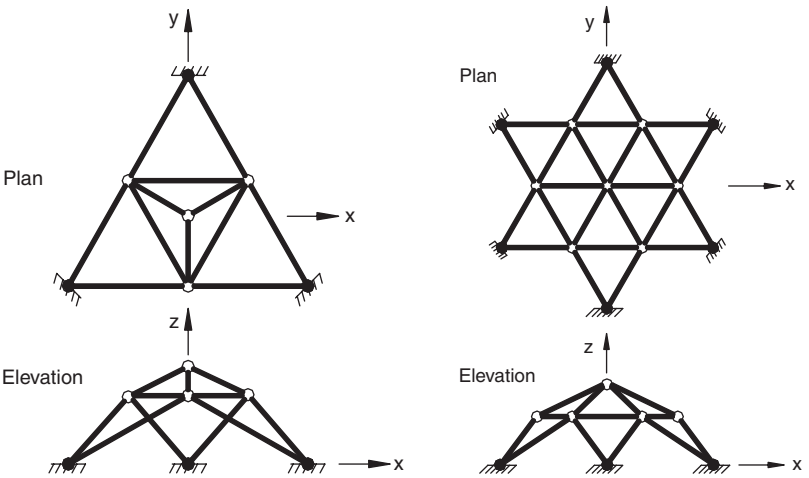

Figure 2. Truss domes with D_n symmetry: (a) D_3 configuration and (b) D_6 configuration [29].

of global bifurcation paths, but also in avoiding possible numerical 'straying' (jumping onto the wrong path).

In tackling the problem of the postbuckling behaviour of an axially compressed cylindrical shell, Wohlever and Healey [28] note that the main computational difficulty is the severe ill-conditioning of the tangent-stiffness matrix, owing to the closely spaced symmetry-breaking bifurcation points on the primary axisymmetric solution branch. They then employ group theory to circumvent this ill-conditioning, by using symmetry-adapted variables (reflecting the symmetry of the various solution paths, and hence also referred to as 'symmetry modes') as the basis vectors for the k mutually independent subspaces $U^{(i)}$ of the original vector space V of the problem. The finite element discretization adopted uses ring elements (to preserve the axisymmetry of the full cylinder), while the displacement fields of the shell (within each ring) are represented by a finite number of Fourier terms along the circumference (the number of terms being chosen to match the deformation mode being investigated). For the vector equilibrium equation

$$\mathbf{f}(\mathbf{u}, \lambda) = \mathbf{0} \tag{5}$$

(where $\mathbf{f}$ is a column-vector function, $\mathbf{u}$ is a displacement vector and λ is a loading parameter), the tangent-stiffness matrix $\mathbf{K} = \mathrm{d}\mathbf{f}/\mathrm{d}\mathbf{u}$, when thus block-diagonalized, leads to the following form of the incremental equations of the Newton–Rapson procedure:

$$\begin{bmatrix} \mathbf{K}^{(1)} & 0 & \cdot & 0 \\ 0 & \mathbf{K}^{(2)} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \mathbf{K}^{(k)} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u}^{(1)} \\ \Delta\mathbf{u}^{(2)} \\ \cdot \\ \Delta\mathbf{u}^{(k)} \end{bmatrix} = \begin{bmatrix} -\mathbf{f}_\lambda \\ 0 \\ \cdot \\ 0 \end{bmatrix} \tag{6}$$

DOI: 10.1002/nme

where $\mathbf{u}^{(i)}$ belongs to the subspace $U^{(i)}$. By reducing the full problem to subproblems of a specific symmetry type within the independent subspaces, a subproblem cannot 'see' solutions of a different symmetry type [28], hence, branch switching due to numerical ill-conditioning is effectively bypassed. It is also evident from Equation (6) that block-diagonalization of the tangent-stiffness matrix affords significant savings in computational effort.

Clearly, the problem of the elastic instability of thin shells is an important one, and the group-theoretic formulation presented by Wohlever and Healey [28] is a major step forward in the pursuit of more effective computational strategies for shells. However, the formulation is applicable only to the geometry of the plain (unstiffened) cylindrical shell. Conical shells and the majority of other shells of revolution, while possessing axisymmetry, will generally lack the additional symmetry of the circular cylinder about the transverse bisector plane (midway between the cylinder ends and perpendicular to the axis of rotation). Despite this, it may still be possible to incorporate group-theoretic adaptations in the associated buckling solution procedures in order to allow exploitation of the original axisymmetry. This is a possible line of study for future work.

Looking at the global bifurcation problem from a more general perspective, and following Ikeda and Murota [27], let us suppose that the non-linear equations of a symmetric multi-dimensional system are represented by an equation of the form of Equation (5). Let us assume the displacement vector $\mathbf{u}$ is an independent variable belonging to an N-dimensional real vector space $V=\Re^N$. Let G be the symmetry group describing the overall symmetry of the system, and g any element belonging to G. If $T(g)$ for $g\in G$ is an $N\times N$ orthogonal matrix of a linear representation of G on $\Re^N$, then Equation (5) is said to be *equivariant* under the action of T if

$$\mathbf{f}(T(g)\mathbf{u},\lambda)=T(g)\mathbf{f}(\mathbf{u},\lambda) \quad \text{for all } g\in G \tag{7}$$

Equivariance of equations to a group indicates that they are invariant under the transformation by *all* elements of the group, and this equivariance is also a property of the tangent-stiffness matrix $\mathbf{K}$ (the Jacobian of $\mathbf{f}$):

$$\mathbf{K}(T(g)\mathbf{u},\lambda)T(g)=T(g)\mathbf{K}(\mathbf{u},\lambda) \quad \text{for all } g\in G \tag{8}$$

If $\mathbf{u}$ is invariant under G, that is, if $T(g)\mathbf{u}=\mathbf{u}$ for all $g\in G$, then Equation (8) becomes

$$\mathbf{K}(\mathbf{u},\lambda)T(g)=T(g)\mathbf{K}(\mathbf{u},\lambda) \quad \text{for all } g\in G \tag{9}$$

For structures with the symmetry of regular polygons, the system of Equation (5) is equivariant to the group D_n (the dihedral group of degree n).

According to group representation theory, the commutativity of $\mathbf{K}$ with the operations of $G(=D_n)$ guarantees the existence of an orthogonal matrix $\mathbf{H}$ which transforms the tangent-stiffness matrix $\mathbf{K}$ into a block-diagonal form $\bar{\mathbf{K}}$:

$$\mathbf{H}^{\mathrm{T}}\mathbf{K}\mathbf{H}=\bar{\mathbf{K}}=\mathrm{diag}[\bar{\mathbf{K}}^l] \tag{10}$$

where $\bar{\mathbf{K}}^l$ $(l=1,2,\ldots,\beta)$ represents the diagonal square blocks making up $\bar{\mathbf{K}}$, β is the total number of these blocks. The $\bar{\mathbf{K}}^l$ are assembled by performing an orthogonal transformation on $\mathbf{K}$, using the appropriate column vectors of the $\mathbf{H}$ matrix (that is, $\mathbf{H}^l$) as the operator:

$$\bar{\mathbf{K}}^l=[\mathbf{H}^l]^{\mathrm{T}}\mathbf{K}\mathbf{H}^l \tag{11}$$

Details of the computation of $\mathbf{H}$ may be seen in the paper by Ikeda and Murota [27]. Each block of $\mathbf{K}$ is associated with a subgroup of D_n expressing the symmetry of the corresponding subspace

DOI: 10.1002/nme

spanned by its column vectors. Instead of working with $\mathbf{K}$ (which may be very large for large-scale problems), tests for singularity are performed independently on the basis of these diagonal blocks $\bar{\mathbf{K}}^l$, which are much smaller in size. In addition, these diagonal blocks are generally not all different from each other, but may feature several pairs of identical matrices, so that one needs only deal with a certain number of distinct representative blocks, implying even lesser analytical effort. To reduce the computational inaccuracies that arise in those situations where eigenvalues are equal or nearly equal to each other, Ikeda and co-workers have subsequently proposed an improved approach [30], which is particularly suited to structures with axisymmetry or the symmetry of regular polygons.

The problem with the foregoing approach is that in order to obtain the diagonal blocks $\bar{\mathbf{K}}^l$, it is necessary to obtain the full tangent-stiffness matrix $\mathbf{K}$ of the problem first, which in itself may require substantial computational effort, particularly in the case of very large problems. Instead of block-diagonalizing $\mathbf{K}$ in the manner indicated above, we may follow the approach of Healey [29], and restrict Equation (5) to the subspace $U^{(i)}$ of G:

$$\mathbf{f}^{(i)}(\mathbf{v}, \lambda) = \mathbf{0} \tag{12}$$

As Healey points out [29], not all solutions of Equation (5) will be solutions of Equations (12), but rather, the reduced problem as represented by Equation (12) will capture only the solution points belonging to the subspace $U^{(i)}$. More importantly, all the solutions of Equation (5) can be generated by solving the subproblem (Equation (12)) for each subspace $U^{(i)}$ $(i = 1, 2, \ldots, k)$. To achieve this, Equation (12) must first be expressed relative to a basis for the subspace in question. Basis vectors are written down using the idempotents (projection operators) of the symmetry group, and all the usual steps of a bifurcation analysis may then be carried out within the independent subspaces.

Let us consider further qualitative insights that group theory may yield on the bifurcation behaviour of a problem. Ikeda and co-workers [26] have studied the bifurcation behaviour of D_n and C_n symmetric structures, and derived a logical sequence or 'hierarchy' for the symmetry-breaking process of bifurcation. One of their observations is that the number of bifurcation paths for such systems is equal to $2n/m$ (twice the ratio of the order of a group to that of a subgroup). Bifurcation paths can undergo further progressive symmetry-breaking bifurcation until they reach a completely asymmetric pattern C_1, and the collection of such successive bifurcations makes up what they term a 'hierarchy' (Figure 3). These concepts are applied to the specific problem of the bifurcation analysis of a rectangular plate [31]. Qualitative considerations of this type allow useful insights to be gained in advance of actual numerical computations.

Aston [32] has considered the problem of symmetry-breaking bifurcation for non-linear equations defined on a real Hilbert space. This has enabled the decomposition of the Hilbert space into orthogonal isotypic components, in turn facilitating the detection and computation of bifurcation points. However, while much of the theoretical treatment is of general applicability and should be noted, the considerations are applied to the gravity wave problem, which lies outside the scope of the present review.

Many engineering structures, including those that are symmetric, have some form of imperfection or another. In their book, Ikeda and Murota [33] consider the bifurcation behaviour of symmetric structures with initial imperfections, and develop group-theoretic solution strategies for these. This is an excellent treatment of the subject, mathematically rigorous and well illustrated with structural examples (truss arches, truss domes, rectangular plates, etc.). The considerations are extended to

DOI: 10.1002/nme

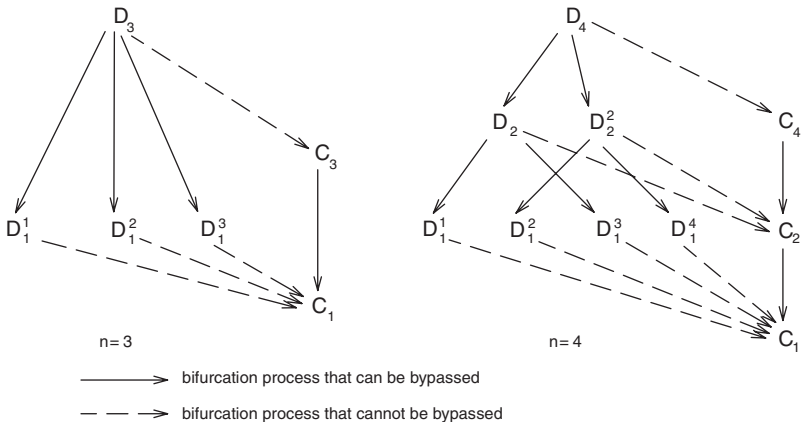

Figure 3. Examples of hierarchies of bifurcation paths of D_n-symmetric structures [26].

the bifurcation behaviour of material test specimens with imperfections (soil, sand and concrete), a most interesting line of application of group-theoretic bifurcation.

4. VIBRATION PROBLEMS

4.1. Problem formulation

Group theory has been extremely successful in solving numerous problems of the vibration of atoms, molecules and crystals [2, 3, 7, 8]. Similar procedures have been applied to computational engineering mechanics, to simplify the vibration analysis of complex mechanical and structural systems. Early applications of group theory to problems of vibration in structural mechanics may be seen in the work of Yang [34] and that of Singh and Mishra [35]. These investigators were able to use concepts of group theory to predict the modes of vibration of symmetric structures.

Large eigenvalue problems (for which the degrees of freedom may be of the order of tens of thousands or even millions) always pose a special challenge from a computational perspective. It is in the analysis of such problems that group-theoretic procedures can be of most benefit. With the eigenvalue analysis of large-scale symmetric skeletal structures in mind, Healey and Treacy [36] have proposed a computational scheme combining group-theoretic ideas and substructuring techniques, to block-diagonalize the structural matrix. Instead of working with the full structural matrix, they use the smaller (and hence more convenient) matrix of the repeating substructure. Let us consider the well-known eigenvalue problem

$$\mathbf{K}\boldsymbol{\varphi} = \omega^2 \mathbf{M}\boldsymbol{\varphi} \tag{13}$$

describing the small free vibrations of a structural–mechanical system. Consistent with the usual notation, $\mathbf{K}$ denotes the $n \times n$ stiffness matrix, $\mathbf{M}$ the $n \times n$ mass matrix, ω a circular frequency and $\boldsymbol{\varphi}$ the corresponding mode shape. According to Healey and Treacy [36], if the system has symmetry, then group representation theory can be employed to assemble an $n \times n$ orthogonal

DOI: 10.1002/nme

matrix $\mathbf{T}$ which transforms $\mathbf{M}$ and $\mathbf{K}$ into block-diagonal form:

$$\mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{T}=\bar{\mathbf{M}} \tag{14a}$$

$$\mathbf{T}^{\mathrm{T}}\mathbf{K}\mathbf{T}=\bar{\mathbf{K}} \tag{14b}$$

These transformations effectively reduce Equation (13) to a series of smaller decoupled eigenvalue problems.

To confirm the validity of Equation (14), let us use the argument of Healey and Treacy [36]. A representation R_j $(j=1,2,\ldots,m;\ m$ is the order of $G)$ for an n-dimensional symmetric problem may be decomposed into its k irreducible representations $R_j^{(i)}$ $(i=1,2,\ldots,k)$ of smaller dimension $\delta^{(i)}$. This decomposition may be achieved via an orthogonal similarity transformation:

$$\mathbf{T}^{\mathrm{T}}\mathbf{R}_j\mathbf{T}=\mathrm{diag}[\mathbf{R}_j^{(1)},\mathbf{R}_j^{(2)},\ldots,\mathbf{R}_j^{(k)}]=\bar{\mathbf{R}}_j \tag{15}$$

where $\mathbf{T}$ is an $n\times n$ orthogonal matrix, each $\mathbf{R}_j^{(i)}$ is an $n^{(i)}\times n^{(i)}$ matrix $(i=1,2,\ldots,k)$ and the sum of the $n^{(i)}$ $(i=1,2,\ldots,k)$ is equal to n. Applying the orthogonal similarity transformation of Equation (15) upon both $\mathbf{M}$ and $\mathbf{K}$, we find that

$$\bar{\mathbf{M}}\bar{\mathbf{R}}_j=\bar{\mathbf{R}}_j\bar{\mathbf{M}},\quad \bar{\mathbf{K}}\bar{\mathbf{R}}_j=\bar{\mathbf{R}}_j\bar{\mathbf{K}}\quad (j=1,2,\ldots,m) \tag{16}$$

This leads us to conclude that $\bar{\mathbf{M}}$ and $\bar{\mathbf{K}}$ must have the same block-diagonal form as the $\bar{\mathbf{R}}_j$:

$$\bar{\mathbf{M}}=\mathrm{diag}[\mathbf{M}^{(1)},\mathbf{M}^{(2)},\ldots,\mathbf{M}^{(k)}] \tag{17a}$$

$$\bar{\mathbf{K}}=\mathrm{diag}[\mathbf{K}^{(1)},\mathbf{K}^{(2)},\ldots,\mathbf{K}^{(k)}] \tag{17b}$$

where $\mathbf{M}^{(i)}$ and $\mathbf{K}^{(i)}$ $(i=1,2,\ldots,k)$ are each $n^{(i)}\times n^{(i)}$ matrices.

The assembly of the orthogonal similarity transformation is equivalent to finding a new system of coordinates, called *symmetry-adapted coordinates* or symmetry modes, for which $\bar{\mathbf{M}}$ and $\bar{\mathbf{K}}$ have the above diagonal form. Symmetry-adapted coordinates may be written down using the respective idempotents. For each subspace $U^{(i)}$ $(i=1,2,\ldots,k)$, there are $n^{(i)}$ symmetry-adapted coordinates or basis vectors associated with $\mathbf{M}^{(i)}$ and $\mathbf{K}^{(i)}$. Let $\mathbf{T}^{(i)}$ denote the $n\times n^{(i)}$ matrix having the symmetry modes $\psi_j^{(i)}$ $(j=1,2,\ldots,n^{(i)})$ as columns:

$$\mathbf{T}^{(i)}=[\psi_1^{(i)}\ \ \psi_2^{(i)}\ \ \cdot\ \ \psi_{n^{(i)}}^{(i)}] \tag{18}$$

Then

$$\mathbf{M}^{(i)}=[\mathbf{T}^{(i)}]^{\mathrm{T}}\mathbf{M}\mathbf{T}^{(i)} \tag{19a}$$

$$\mathbf{K}^{(i)}=[\mathbf{T}^{(i)}]^{\mathrm{T}}\mathbf{K}\mathbf{T}^{(i)} \tag{19b}$$

In their paper, Healey and Treacy [36] proceed to outline an algorithm based on the repeating substructure of the full structure, which does away with the need to first assemble the global matrices $\mathbf{M}$ and $\mathbf{K}$ in order to evaluate the blocks $\mathbf{M}^{(i)}$ and $\mathbf{K}^{(i)}$. This improvement is particularly advantageous for large-scale problems. Numerical examples considered include a dish antenna with C_{6v} symmetry, and space trusses with D_3 and C_{3h} symmetries. Of course one has to be able to correctly identify the repeating substructure, and one still needs to compute the matrices for the

DOI: 10.1002/nme

substructure. Techniques that completely avoid having to obtain **M** and **K** (whether for the full structure or for a substructure) will shortly be illustrated. These generate $\mathbf{M}^{(i)}$ and $\mathbf{K}^{(i)}$ directly, not via transformations of the type shown in Equations (19).

Zlokovic [14] has developed effective group-theoretic procedures for problems of the vibration analysis of symmetric structural systems (such as lumped-parameter models of beams), among others. The approach involves the decomposition of the vector space of the normal variables of a problem into a number of independent subspaces spanned by symmetry-adapted variables, permitting quantities of interest to be separately obtained for each subspace through the solution of a system of equations with only a fraction of the number of unknowns in the original problem. The overall result is a substantial reduction in computational effort, in comparison with conventional procedures.

Other investigators have adopted similar approaches in formulating the vibration problem for various symmetric structures. The author has presented specific formulations for the vibration of high-tension cable nets [37], grid-mass systems [38] and isotropic plates [39]. The same techniques have also permitted useful insight to be gained on the vibration properties (symmetries of modes, repeated frequencies, etc.) of space structures of complex symmetry (such as layered space grids) [40], before any detailed numerical computations are undertaken. Mohan and Pratap [41] use a group-theoretic formulation to tackle the free vibration problem of engineering ducts of polygonal cross-section, and also extract useful *a priori* information on modes and frequencies (clustering of modes, repeated roots).

To illustrate the group-theoretic computational steps for obtaining actual frequencies and mode shapes, we will consider two particular examples: (i) the vibration of a spring–mass system; (ii) the vibration of a cable net. The first example illustrates how seemingly unsymmetrical physical systems can be transformed into symmetric configurations amenable to analysis by group-theoretic techniques. In both examples, we do not require the assembly of the global stiffness matrix **K** first (or the global flexibility matrix **A** in the case of the cable net, where a flexibility formulation is adopted). Instead, symmetry-adapted stiffness matrices $\mathbf{K}^{(i)}$ (or symmetry-adapted flexibility matrices $\mathbf{A}^{(i)}$ in the case of the cable net) are directly computed within the independent subspaces $U^{(i)}$ of the problem. As the models adopted are lumped-parameter systems, no computations for the $\mathbf{M}^{(i)}$ will actually be necessary; these will simply be written down, but following group-theoretic rules of course.

4.2. Spring–mass mechanical system

Techniques for simplifying the calculation of eigenvalues of matrices with certain special patterns, such matrices representing symmetric graph models of various structures, have been demonstrated by Kaveh and co-investigators [42]. The dynamic behaviour of complex engineering systems is often modelled by means of spring–mass assemblies of appropriate mass, stiffness and damping values. Certain configurations of spring–mass systems, while themselves not physically symmetric, may be transformed into equivalent symmetric models that preserve all the graphical connectivities between masses and springs, allowing group theory to be employed to simplify the extraction of eigenvalues for the equivalent system. In more recent considerations of the vibration of such systems, Kaveh and Nikbakht [43] successfully bring together techniques of graphs and group theory, and employ the group-theoretic approach to simplify the calculation of eigenvalues. Their procedure involves assembly of the full stiffness matrix of the entire system, which is then block-diagonalized via a transformation matrix.

DOI: 10.1002/nme

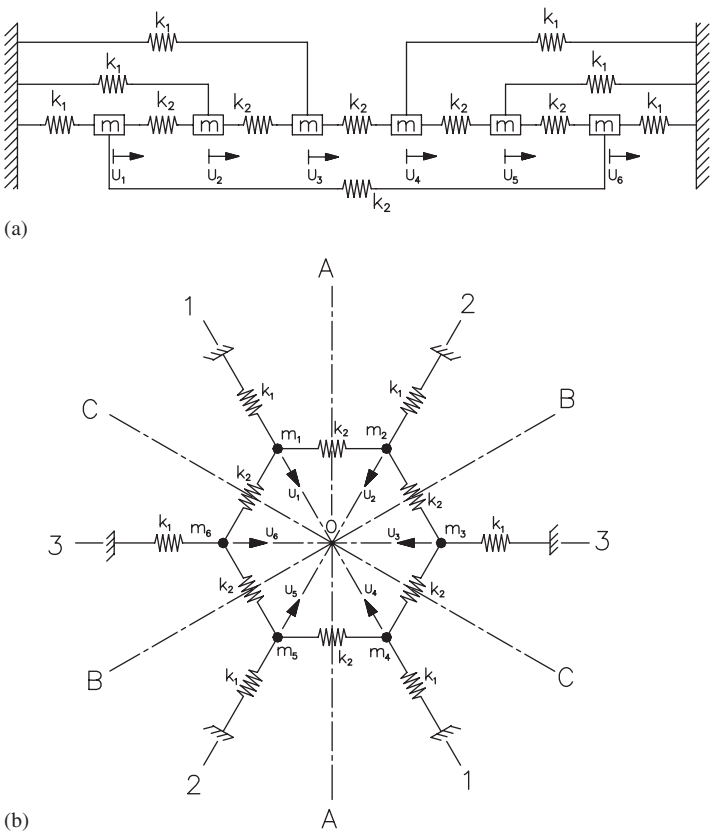

Figure 4. A 6 d.o.f. spring–mass system: (a) original configuration and (b) transformed configuration with C_{6v} symmetry.

An alternative approach to similar problems has been adopted by the author [44]. This involves formulating the eigenvalue problem for each subspace directly, without the need for first assembling the full stiffness matrix for the entire system. This may have computational advantages in the case of very large problems. The example below (drawn from Reference [44]) illustrates the procedure.

A spring–mass system with six degrees of freedom is shown in Figure 4(a). The masses are all equal ($m_i = m$ for $i = 1, 2, \ldots, 6$). The equivalent symmetric system is shown in Figure 4(b). This configuration has a centre of symmetry O, and the symmetry elements: e (identity); c_6 and c_6^{-1} (clockwise and anticlockwise rotations, respectively, of $2\pi/6$ about the centre of symmetry O); c_3 and c_3^{-1} (clockwise and anticlockwise rotations, respectively, of $2\pi/3$ about the centre of symmetry O); c_2 (rotation of π about the centre of symmetry O); σ_a, σ_b, σ_c, σ_1, σ_2 and σ_3 (reflections in the vertical planes A–A, B–B, C–C, 1–1, 2–2 and 3–3, respectively). The configuration of Figure 4(b) therefore belongs to the symmetry group C_{6v} of order 12. The positive direction of the freedoms u_1, u_2, u_3, u_4, u_5 and u_6 maps into the direction towards the centre of symmetry O.

We follow the general approach outlined in Section 2, and begin by writing down the idempotents of symmetry group C_{6v}, using Equation (3). There are six of these, corresponding to the subspaces

DOI: 10.1002/nme

$U^{(1)}$, $U^{(2)}$, $U^{(3)}$, $U^{(4)}$, $U^{(5)}$ and $U^{(6)}$ of the problem. The results are:

$$P^{(1)} = \tfrac{1}{12}(e+c_6+c_6^{-1}+c_3+c_3^{-1}+c_2+\sigma_a+\sigma_b+\sigma_c+\sigma_1+\sigma_2+\sigma_3) \tag{20a}$$

$$P^{(2)} = \tfrac{1}{12}(e+c_6+c_6^{-1}+c_3+c_3^{-1}+c_2-\sigma_a-\sigma_b-\sigma_c-\sigma_1-\sigma_2-\sigma_3) \tag{20b}$$

$$P^{(3)} = \tfrac{1}{12}(e-c_6-c_6^{-1}+c_3+c_3^{-1}-c_2+\sigma_a+\sigma_b+\sigma_c-\sigma_1-\sigma_2-\sigma_3) \tag{20c}$$

$$P^{(4)} = \tfrac{1}{12}(e-c_6-c_6^{-1}+c_3+c_3^{-1}-c_2-\sigma_a-\sigma_b-\sigma_c+\sigma_1+\sigma_2+\sigma_3) \tag{20d}$$

$$P^{(5)} = \tfrac{1}{6}(2e+c_6+c_6^{-1}-c_3-c_3^{-1}-2c_2) \tag{20e}$$

$$P^{(6)} = \tfrac{1}{6}(2e-c_6-c_6^{-1}-c_3-c_3^{-1}+2c_2) \tag{20f}$$

If we apply the first idempotent $P^{(1)}$ to all the six freedoms u_i $(i=1,2,\ldots,6)$ of the configuration of Figure 4(b), we obtain six symmetry-adapted freedoms for the first subspace, not all of which are independent. Selecting a set of linearly independent symmetry-adapted freedoms as the basis vectors for subspace $U^{(1)}$, and repeating the procedure for all the other subspaces (each time using the correct idempotent for the subspace in question), we obtain the following results for the basis vectors of the subspaces:

Subspace $U^{(1)}$

$$\Phi_1^{(1)} = u_1+u_2+u_3+u_4+u_5+u_6 \tag{21}$$

Subspace $U^{(4)}$

$$\Phi_1^{(4)} = u_1-u_2+u_3-u_4+u_5-u_6 \tag{22}$$

Subspace $U^{(5)}$

$$\Phi_1^{(5)} = u_1+\tfrac{1}{2}u_2-\tfrac{1}{2}u_3-u_4-\tfrac{1}{2}u_5+\tfrac{1}{2}u_6 \tag{23a}$$

$$\Phi_2^{(5)} = \tfrac{1}{2}u_1+u_2+\tfrac{1}{2}u_3-\tfrac{1}{2}u_4-u_5-\tfrac{1}{2}u_6 \tag{23b}$$

Subspace $U^{(6)}$

$$\Phi_1^{(6)} = u_1-\tfrac{1}{2}u_2-\tfrac{1}{2}u_3+u_4-\tfrac{1}{2}u_5-\tfrac{1}{2}u_6 \tag{24a}$$

$$\Phi_2^{(6)} = -\tfrac{1}{2}u_1+u_2-\tfrac{1}{2}u_3-\tfrac{1}{2}u_4+u_5-\tfrac{1}{2}u_6 \tag{24b}$$

We observe from the above results that subspaces $U^{(1)}$ and $U^{(4)}$ are 1-dimensional (that is, spanned by one basis vector each), subspaces $U^{(5)}$ and $U^{(6)}$ are 2-dimensional, while subspaces $U^{(2)}$ and $U^{(3)}$ are null spaces. We have thus decomposed the original 6-dimensional vector space of the problem into four independent subspaces of smaller dimensions, spanned by the above sets of symmetry-adapted freedoms. Each subspace is associated with a specific symmetry type.

For subspace $U^{(1)}$, if we apply a unit positive displacement simultaneously on each mass m_i $(i=1,2,\ldots,6)$ in accordance with the coordinates of the basis vector $\Phi_1^{(1)}$ (Equation (21)), we see that the ensuing restoring force on each mass is equal to k_1. Thus,

$$\mathbf{K^{(1)}} = k_{11}^{(1)} = [k_1], \quad \mathbf{M^{(1)}} = m_{11}^{(1)} = [m] \tag{25}$$

DOI: 10.1002/nme

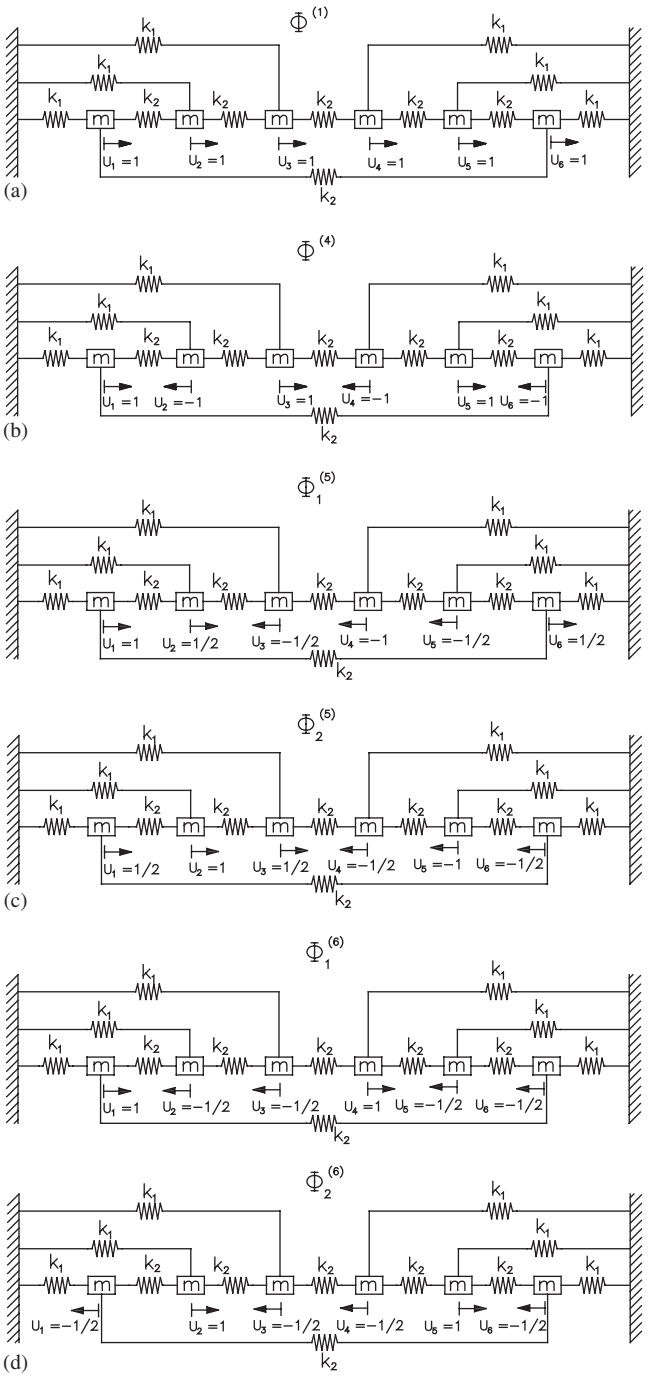

Figure 5. Application of displacement components of the basis vectors for the spring–mass model: (a) subspace $U^{(1)}$; (b) subspace $U^{(4)}$; (c) subspace $U^{(5)}$; and (d) subspace $U^{(6)}$.

DOI: 10.1002/nme

This leads to the first-degree characteristic equation

$$k_1 - \omega^2 m = 0 \tag{26}$$

and the solution

$$\omega^2 = \frac{k_1}{m} \tag{27}$$

For subspace $U^{(4)}$, applying a unit displacement simultaneously on each mass, in accordance with the coordinates of the basis vector $\Phi_1^{(4)}$ (Equation (22)) as shown in Figure 5(b), results in a restoring force of (k_1+4k_2) on each mass. Thus,

$$\mathbf{K^{(4)}} = k_{11}^{(4)} = [k_1 + 4k_2], \quad \mathbf{M^{(4)}} = m_{11}^{(4)} = [m] \tag{28}$$

which leads to the first-degree characteristic equation

$$(k_1 + 4k_2) - \omega^2 m = 0 \tag{29}$$

and the solution

$$\omega^2 = \frac{k_1 + 4k_2}{m} \tag{30}$$

For subspace $U^{(5)}$, the restoring forces on each of the masses m_i $(i = 1, 2, \ldots, 6)$, resulting from the application of the displacement components making up the basis vectors $\Phi_1^{(5)}$ and $\Phi_2^{(5)}$ (Equations (23)) as shown in Figure 5(c), are summarized in Table II. Similarly, for subspace $U^{(6)}$, the restoring forces on each of the masses m_i, resulting from the application of the displacement components making up the basis vectors $\Phi_1^{(6)}$ and $\Phi_2^{(6)}$ (Equations (24)) as shown in Figure 5(d), are summarized in Table III.

Consider subspace $U^{(5)}$ first. For the application of either $\Phi_1^{(5)}$ or $\Phi_2^{(5)}$, it is noted (from Table II) that all restoring forces are proportional to the u value prescribed to the mass in question. Taking unit positive value of u (i.e. $u = +1$), the corresponding restoring force is $(k_1 + k_2)$ in either case. Therefore,

$$\begin{aligned} k_{11}^{(5)} &= (k_1 + k_2), \quad k_{12}^{(5)} = 0 \\ k_{21}^{(5)} &= 0, \quad k_{22}^{(5)} = (k_1 + k_2) \end{aligned} \tag{31}$$

Table II. Restoring forces due to the application of the components of $\Phi_1^{(5)}$ and $\Phi_2^{(5)}$ for the spring–mass system.

	$\Phi_1^{(5)}$		$\Phi_2^{(5)}$	
Mass	Displacement	Restoring force	Displacement	Restoring force
$m_1 = m$	$u_1 = +1.0$	$+1.0(k_1 + k_2)$	$u_1 = +0.5$	$+0.5(k_1 + k_2)$
$m_2 = m$	$u_2 = +0.5$	$+0.5(k_1 + k_2)$	$u_2 = +1.0$	$+1.0(k_1 + k_2)$
$m_3 = m$	$u_3 = -0.5$	$-0.5(k_1 + k_2)$	$u_3 = +0.5$	$+0.5(k_1 + k_2)$
$m_4 = m$	$u_4 = -1.0$	$-1.0(k_1 + k_2)$	$u_4 = -0.5$	$-0.5(k_1 + k_2)$
$m_5 = m$	$u_5 = -0.5$	$-0.5(k_1 + k_2)$	$u_5 = -1.0$	$-1.0(k_1 + k_2)$
$m_6 = m$	$u_6 = +0.5$	$+0.5(k_1 + k_2)$	$u_6 = -0.5$	$-0.5(k_1 + k_2)$

DOI: 10.1002/nme

Table III. Restoring forces due to the application of the components of $\Phi_1^{(6)}$ and $\Phi_2^{(6)}$ for the spring–mass system.

	$\Phi_1^{(6)}$		$\Phi_2^{(6)}$	
Mass	Displacement	Restoring force	Displacement	Restoring force
$m_1=m$	$u_1=+1.0$	$+1.0(k_1+3k_2)$	$u_1=-0.5$	$-0.5(k_1+3k_2)$
$m_2=m$	$u_2=-0.5$	$-0.5(k_1+3k_2)$	$u_2=+1.0$	$+1.0(k_1+3k_2)$
$m_3=m$	$u_3=-0.5$	$-0.5(k_1+3k_2)$	$u_3=-0.5$	$-0.5(k_1+3k_2)$
$m_4=m$	$u_4=+1.0$	$+1.0(k_1+3k_2)$	$u_4=-0.5$	$-0.5(k_1+3k_2)$
$m_5=m$	$u_5=-0.5$	$-0.5(k_1+3k_2)$	$u_5=+1.0$	$+1.0(k_1+3k_2)$
$m_6=m$	$u_6=-0.5$	$-0.5(k_1+3k_2)$	$u_6=-0.5$	$-0.5(k_1+3k_2)$

The symmetry-adapted stiffness matrix $\mathbf{K}^{(5)}$ is therefore (corresponding mass matrix also given)

$$\mathbf{K}^{(5)}=\begin{bmatrix}(k_1+k_2) & 0\\ 0 & (k_1+k_2)\end{bmatrix},\quad \mathbf{M}^{(5)}=\begin{bmatrix}m & 0\\ 0 & m\end{bmatrix} \tag{32}$$

This leads to two uncoupled first-degree characteristic equations

$$(k_1+k_2)-\omega^2 m=0\quad\text{(twice)} \tag{33}$$

giving the solutions

$$(\omega^2)_1^{(5)}=(\omega^2)_2^{(5)}=\frac{k_1+k_2}{m} \tag{34}$$

Similarly, for subspace $U^{(6)}$, for the application of either $\Phi_1^{(6)}$ or $\Phi_2^{(6)}$, it is noted (from Table III) that all restoring forces are also proportional to the u value prescribed to the mass in question. Taking unit positive value of u (i.e. $u=+1$), the corresponding restoring force is (k_1+3k_2) in either case. Hence,

$$\begin{aligned} k_{11}^{(6)}&=(k_1+3k_2),\quad k_{12}^{(6)}=0\\ k_{21}^{(6)}&=0,\quad k_{22}^{(6)}=(k_1+3k_2)\end{aligned} \tag{35}$$

The symmetry-adapted stiffness matrix $\mathbf{K}^{(6)}$ is therefore (corresponding mass matrix also given)

$$\mathbf{K}^{(6)}=\begin{bmatrix}(k_1+3k_2) & 0\\ 0 & (k_1+3k_2)\end{bmatrix},\quad \mathbf{M}^{(6)}=\begin{bmatrix}m & 0\\ 0 & m\end{bmatrix} \tag{36}$$

leading to two uncoupled first-degree characteristic equations

$$(k_1+3k_2)-\omega^2 m=0\quad\text{(twice)} \tag{37}$$

with the solutions

$$(\omega^2)_1^{(6)}=(\omega^2)_2^{(6)}=\frac{k_1+3k_2}{m} \tag{38}$$

DOI: 10.1002/nme

In summary, subspaces $U^{(1)}$ and $U^{(4)}$ yielded one eigenvalue each; subspaces $U^{(5)}$ and $U^{(6)}$ each yielded two equal eigenvalues (doubly repeating roots). Putting the solutions together, the six natural circular frequencies of the original system (in ascending order) are as follows:

$$\omega_1^2=\frac{k_1}{m},\quad \omega_2^2=\omega_3^2=\frac{k_1+k_2}{m},\quad \omega_4^2=\omega_5^2=\frac{k_1+3k_2}{m},\quad \omega_6^2=\frac{k_1+4k_2}{m} \tag{39}$$

Instead of having to expand a 6×6 determinant and solve a sixth-degree characteristic polynomial as yielded by conventional considerations, the group-theoretic approach required the solution of only four separate *first-degree* characteristic equations, yielding all six eigenvalues (two distinct roots and two pairs of repeated roots) of the original problem. Here we see that the computational effort of the group-theoretic procedure is less than 10% of that associated with conventional analysis. Spring–mass models of other symmetry groups may be seen in Reference [44].

4.3. High-tension cable net

The previous example was based on the stiffness formulation of the vibration problem. To illustrate the group-theoretic flexibility formulation, let us consider a shallow non-planar cable net formed by two families of highly tensioned cables intersecting perpendicularly on the xy horizontal projection (Figure 6). The net has 16 nodes or cable intersections, numbered as shown in the figure. This particular example is taken from Reference [37].

Clearly, the configuration has the symmetry of a square, and therefore belongs to the symmetry group C_{4v}. The four reflection planes are labelled $x, y, 1$ and 2, while the centre of symmetry is denoted by O. In this example, the horizontal spacing between cables is taken as constant and equal to a. The horizontal component of the cable prestress force, denoted as T_1 or T_2 in the

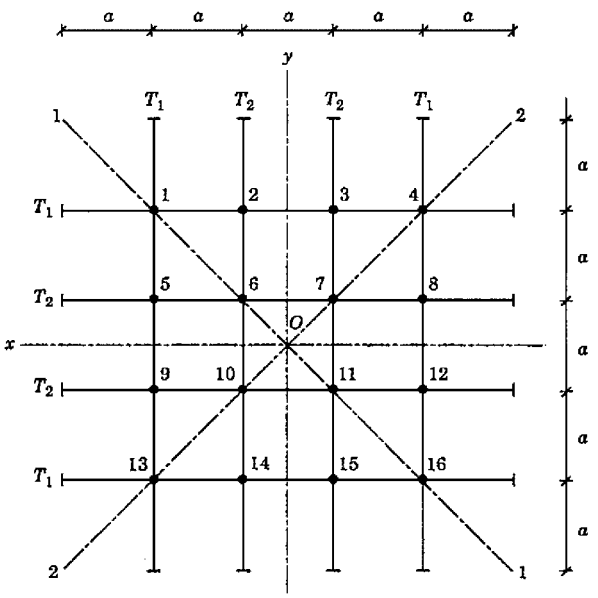

Figure 6. Horizontal projection of 16-node cable net, showing adopted node numbering, vertical planes of symmetry, and pattern of prestressing.

DOI: 10.1002/nme

figure, is assumed to be constant over the entire length of the cable. The T_1 and T_2 pattern (that is, the loading pattern) conforms to the C_{4v} symmetry of the cable layout. We require to evaluate the free vibration response of the system, where the degrees of freedom are the small transverse motions of masses lumped at the cable intersections.

4.3.1. Symmetry-adapted functions. The 16-dimensional vector space of the problem can be decomposed into five independent subspaces spanned by symmetry-adapted variables. The five idempotents of group C_{4v}, corresponding to subspaces $U^{(1)}$, $U^{(2)}$, $U^{(3)}$, $U^{(4)}$ and $U^{(5)}$ of the problem, are generated with the help of Equation (3). The results are as follows:

$$P^{(1)} = \tfrac{1}{8}(e + c_4 + c_4^{-1} + c_2 + \sigma_x + \sigma_y + \sigma_1 + \sigma_2) \tag{40a}$$

$$P^{(2)} = \tfrac{1}{8}(e + c_4 + c_4^{-1} + c_2 - \sigma_x - \sigma_y - \sigma_1 - \sigma_2) \tag{40b}$$

$$P^{(3)} = \tfrac{1}{8}(e - c_4 - c_4^{-1} + c_2 + \sigma_x + \sigma_y - \sigma_1 - \sigma_2) \tag{40c}$$

$$P^{(4)} = \tfrac{1}{8}(e - c_4 - c_4^{-1} + c_2 - \sigma_x - \sigma_y + \sigma_1 + \sigma_2) \tag{40d}$$

$$P^{(5)} = \tfrac{1}{2}(e - c_2) \tag{40e}$$

To obtain the symmetry-adapted positional functions for a given subspace $U^{(i)}$ $(i = 1, 2, \ldots, 5)$, we first apply the corresponding idempotent $P^{(i)}$ as an operator on each positional function ϕ_j $(j = 1, 2, \ldots, 16)$ of the problem, and then select a linearly independent set of the symmetry-adapted functions as the basis vectors spanning the subspace. The results for our problem are as follows:

Subspace $U^{(1)}$

$$\Phi_1^{(1)} = \phi_1 + \phi_4 + \phi_{13} + \phi_{16} \tag{41a}$$

$$\Phi_2^{(1)} = \phi_2 + \phi_3 + \phi_5 + \phi_8 + \phi_9 + \phi_{12} + \phi_{14} + \phi_{15} \tag{41b}$$

$$\Phi_3^{(1)} = \phi_6 + \phi_7 + \phi_{10} + \phi_{11} \tag{41c}$$

Subspace $U^{(2)}$

$$\Phi_1^{(2)} = \phi_2 - \phi_3 - \phi_5 + \phi_8 + \phi_9 - \phi_{12} - \phi_{14} + \phi_{15} \tag{42}$$

Subspace $U^{(3)}$

$$\Phi_1^{(3)} = \phi_2 + \phi_3 - \phi_5 - \phi_8 - \phi_9 - \phi_{12} + \phi_{14} + \phi_{15} \tag{43}$$

Subspace $U^{(4)}$

$$\Phi_1^{(4)} = \phi_1 - \phi_4 - \phi_{13} + \phi_{16} \tag{44a}$$

$$\Phi_2^{(4)} = \phi_2 - \phi_3 + \phi_5 - \phi_8 - \phi_9 + \phi_{12} - \phi_{14} + \phi_{15} \tag{44b}$$

$$\Phi_3^{(4)} = \phi_6 - \phi_7 - \phi_{10} + \phi_{11} \tag{44c}$$

DOI: 10.1002/nme

Subspace $U^{(5)}$

$$\Phi_1^{(5)} = \phi_1 - \phi_{16} \tag{45a}$$

$$\Phi_2^{(5)} = \phi_2 - \phi_{15} \tag{45b}$$

$$\Phi_3^{(5)} = \phi_3 - \phi_{14} \tag{45c}$$

$$\Phi_4^{(5)} = \phi_4 - \phi_{13} \tag{45d}$$

$$\Phi_5^{(5)} = \phi_5 - \phi_{12} \tag{45e}$$

$$\Phi_6^{(5)} = \phi_6 - \phi_{11} \tag{45f}$$

$$\Phi_7^{(5)} = \phi_7 - \phi_{10} \tag{45g}$$

$$\Phi_8^{(5)} = \phi_8 - \phi_9 \tag{45h}$$

We can further decompose the 8-dimensional subspace $U^{(5)}$ into two 4-dimensional subspaces $U^{(51)}$ and $U^{(52)}$, spanned by new basis vectors obtained by linearly combining the vectors of Equation (45) in such a way as to form two orthogonal sets:

Subspace $U^{(51)}$

$$\Phi_1^{(51)} = \Phi_1^{(5)} = \phi_1 - \phi_{16} \tag{46a}$$

$$\Phi_2^{(51)} = \Phi_6^{(5)} = \phi_6 - \phi_{11} \tag{46b}$$

$$\Phi_3^{(51)} = \Phi_3^{(5)} - \Phi_8^{(5)} = \phi_3 - \phi_8 + \phi_9 - \phi_{14} \tag{46c}$$

$$\Phi_4^{(51)} = \Phi_2^{(5)} + \Phi_5^{(5)} = \phi_2 + \phi_5 - \phi_{12} - \phi_{15} \tag{46d}$$

Subspace $U^{(52)}$

$$\Phi_1^{(52)} = \Phi_4^{(5)} = \phi_4 - \phi_{13} \tag{47a}$$

$$\Phi_2^{(52)} = \Phi_7^{(5)} = \phi_7 - \phi_{10} \tag{47b}$$

$$\Phi_3^{(52)} = \Phi_3^{(5)} + \Phi_8^{(5)} = \phi_3 + \phi_8 - \phi_9 - \phi_{14} \tag{47c}$$

$$\Phi_4^{(52)} = \Phi_2^{(5)} - \Phi_5^{(5)} = \phi_2 - \phi_5 + \phi_{12} - \phi_{15} \tag{47d}$$

The orthogonality of the two vector sets (Equations (46) and (47)) may be seen in Figure 7, diagrams (e) and (f), in which solid and dashed lines are shown at the centre of the sketches, to indicate the symmetry and antisymmetry planes, respectively, of the configurations of vector components. The orientations of symmetry and antisymmetry planes in diagram (e) (corresponding to subspace $U^{(51)}$) are perpendicular to their counterparts in diagram (f) (corresponding to subspace $U^{(52)}$), clearly indicating that the two sets of vectors are orthogonal.

The decomposition of subspace $U^{(5)}$ into $U^{(51)}$ and $U^{(52)}$ may be achieved directly by splitting idempotent $P^{(5)}$ (Equation (40e)) into two components $P^{(51)}$ and $P^{(52)}$, such that

DOI: 10.1002/nme

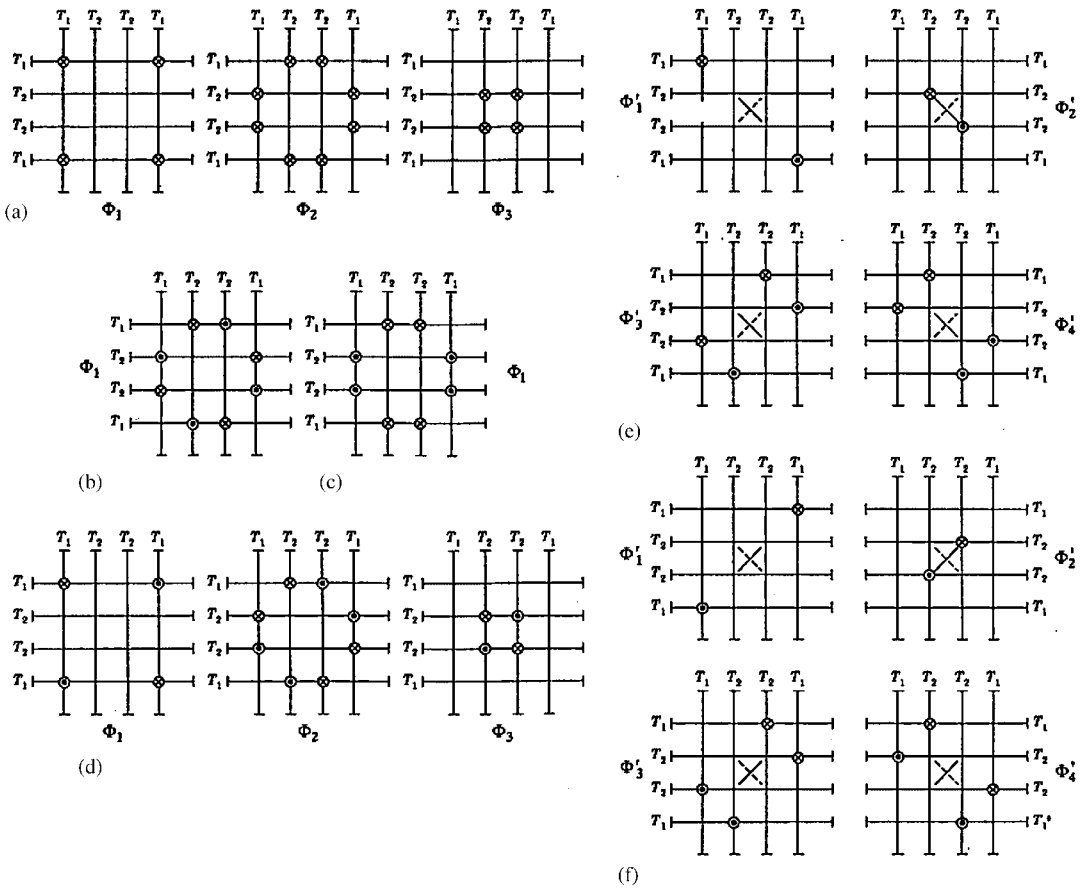

Figure 7. Unit vertical forces applied in accordance with the coordinates of the basis vectors for the 16-node cable net: (a) subspace $U^{(1)}$; (b) subspace $U^{(2)}$; (c) subspace $U^{(3)}$; (d) subspace $U^{(4)}$; (e) subspace $U^{(51)}$; and (f) subspace $U^{(52)}$.

$P^{(51)}+P^{(52)}=P^{(5)}$, $P^{(51)}P^{(51)}=P^{(51)}$, $P^{(52)}P^{(52)}=P^{(52)}$ and $P^{(51)}P^{(52)}=0$. These conditions are satisfied by the operators

$$P^{(51)}=\tfrac{1}{4}(e-c_2+\sigma_1-\sigma_2) \tag{48a}$$

$$P^{(52)}=\tfrac{1}{4}(e-c_2-\sigma_1+\sigma_2) \tag{48b}$$

which may also be seen to be each orthogonal to the idempotents $p^{(i)}$ $(i=1,2,3,4)$ given by Equations (40a)–(40d). Thus, in our example, applying $P^{(51)}$ and $P^{(52)}$ in turn as operators on each positional function ϕ_j $(j=1,2,\ldots,16)$ generates the basis vectors for subspaces $U^{(51)}$ and U_{52} automatically.

At this stage, we note that the 16-dimensional vector space of the original problem has been decomposed into a total of six r-dimensional subspaces, where $r=\{3,1,1,3,4,4\}$ for subspaces $U^{(1)}$, $U^{(2)}$, $U^{(3)}$, $U^{(4)}$, $U^{(51)}$ and $U^{(52)}$, respectively. We already have a good sense of the computational gains to be reaped, even before the real calculations begin.

DOI: 10.1002/nme

4.3.2. Symmetry-adapted flexibility matrices. For each subspace of the problem, we apply unit vertical forces upon the cable-net nodes in accordance with the coordinates of the respective basis vectors, as shown in Figure 7. For a given subspace, each basis vector (with its set of unit vertical forces) is plotted on a separate diagram, for clarity. The notations $\otimes$ and $\odot$ in these diagrams denote positive (i.e. downward) and negative (i.e. upward) unit forces, respectively. The numbering of the nodes is not shown (to avoid cluttering the diagrams), but the diagrams have been drawn with exactly the same orientation as in Figure 6, to which reference may be made for the node numbering.

When all the r basis-vector plots of a given subspace (shown in separate diagrams in Figure 7) are superimposed on one diagram, any nodes that remain 'unaffiliated' (that is, not associated with any one of the r basis vectors of the subspace) imply *stationary* points. Thus, subspaces $U^{(2)}$ and $U^{(3)}$ are each associated with eight stationary nodes (the points lying on *either* principal diagonals of the net), while subspaces $U^{(51)}$ and $U^{(52)}$ are each associated with the occurrence of 4 stationary nodes (the points on one of the two principal diagonals of the net).

From the plots of Figure 7, we make an important observation. The set of basis vectors for subspace $U^{(51)}$ is physically indistinguishable from that for subspace $U^{(52)}$. For every vector in $U^{(51)}$, there is one exactly identical to it in $U^{(52)}$, except for orientation (compare Figures 7(e) and 7(f)). Now the physical properties of a physical system, such as natural frequencies and modes of deformation, are clearly not affected by its orientation in space. Therefore, a solution for any required physical properties based on subspace $U^{(51)}$ must be identical to that based on subspace $U^{(52)}$. These subspaces will yield identical sets of eigenvalues (doubly repeating roots of the original problem). Hence, we need only consider one of them. In the present example, subspace $U^{(51)}$ will be selected.

Consider any one of the subspaces of the problem, such a subspace being spanned by r basis vectors. Let a_{ij} $(i=1,2,\ldots,r;\ j=1,2,\ldots,r)$ be the vertical-displacement magnitude at *any* of the nodes of the basis vector Φ_i, due to unit vertical forces applied at *all* the nodes of the basis vector Φ_j. The condition of vertical equilibrium at each of the r sets of nodes (corresponding to the r basis vectors of the subspace) leads to r simultaneous equations in the r deflection unknowns $\{a_{1j}, a_{2j}, \ldots, a_{rj}\}$, which may be expressed as

$$\begin{bmatrix} b_{11} & b_{12} & \cdot & b_{1r} \\ b_{21} & b_{22} & \cdot & b_{2r} \\ \cdot & \cdot & \cdot & \cdot \\ b_{r1} & b_{r2} & \cdot & b_{rr} \end{bmatrix} \begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ a_{rj} \end{bmatrix} = \begin{bmatrix} \delta_{1j} \\ \delta_{2j} \\ \cdot \\ \delta_{rj} \end{bmatrix} \tag{49}$$

for $j=1,2,\ldots,r;$. For the right-hand side, $\delta_{ij}=1$ if $i=j$, and $\delta_{ij}=0$ if $i\neq j$. We may write this equation as

$$\mathbf{B}^{(l)}\mathbf{A}_j = \boldsymbol{\delta}_j \tag{50}$$

where $\mathbf{B}^{(l)}$ is an $r\times r$ matrix corresponding to subspace $U^{(l)}$, $\boldsymbol{\delta}_j$ is an $r\times 1$ column vector consisting of a '1' at row j and 'zeros' everywhere else, $\mathbf{A}_j$ is the $r\times 1$ column vector of the deflections

DOI: 10.1002/nme

corresponding to the application of unit vertical forces at each of the nodes of Φ_j. These deflections are, in fact, the flexibility coefficients for the subspace.

Rearranging Equation (50) then yields the column vectors $\mathbf{A}_j$ of the flexibility matrix for subspace $U^{(l)}$ (which we can now write as $\mathbf{A}_j^{(l)}$ to denote that they pertain to subspace $U^{(l)}$); when the $\mathbf{A}_j^{(l)}$ are put together, they form the full flexibility matrix $\mathbf{A}^{(l)}$ for subspace $U^{(l)}$:

$$\mathbf{A}_j^{(l)} = [\mathbf{B}^{(l)}]^{-1}\boldsymbol{\delta}_j, \quad j = 1, 2, \ldots, r \tag{51a}$$

$$\mathbf{A}^{(l)} = [\mathbf{A}_1^{(l)} | \mathbf{A}_2^{(l)} | \ldots | \mathbf{A}_r^{(l)}] \tag{51b}$$

The $\mathbf{B}^{(l)}$ matrices corresponding to the various subspaces of the problem may readily be evaluated. The results are as follows:

Subspace $U^{(1)}$

$$\mathbf{B}^{(1)} = \begin{bmatrix} \frac{4T_1}{a} & -\frac{2T_1}{a} & 0 \\ -\frac{T_1}{a} & \frac{T_1+2T_2}{a} & -\frac{T_2}{a} \\ 0 & -\frac{2T_2}{a} & \frac{2T_2}{a} \end{bmatrix} \tag{52}$$

Subspace $U^{(2)}$ *and* $U^{(3)}$

$$\mathbf{B}^{(2)} = \begin{bmatrix} \frac{3T_1+2T_2}{a} \end{bmatrix} \tag{53}$$

$$\mathbf{B}^{(3)} = \begin{bmatrix} \frac{T_1+2T_2}{a} \end{bmatrix} \tag{54}$$

Subspace $U^{(4)}$

$$\mathbf{B}^{(4)} = \begin{bmatrix} \frac{4T_1}{a} & -\frac{2T_1}{a} & 0 \\ -\frac{T_1}{a} & \frac{3T_1+2T_2}{a} & -\frac{T_2}{a} \\ 0 & -\frac{2T_2}{a} & \frac{6T_2}{a} \end{bmatrix} \tag{55}$$

Subspace $U^{(51)}$

$$\mathbf{B}^{(51)} = \begin{bmatrix} \frac{4T_1}{a} & 0 & 0 & -\frac{2T_1}{a} \\ 0 & \frac{4T_2}{a} & 0 & -\frac{2T_2}{a} \\ 0 & 0 & \frac{2(T_1+T_2)}{a} & -\frac{T_1}{a} \\ -\frac{T_1}{a} & -\frac{T_2}{a} & -\frac{T_1}{a} & \frac{2(T_1+T_2)}{a} \end{bmatrix} \tag{56}$$

DOI: 10.1002/nme

4.3.3. Subspace mass matrices. We adopt a system of concentrated masses positioned at the nodes of the cable net—such masses may be actual or lumped parameters of a distributed-mass system—and arranged consistently with the general symmetry of the cable configuration. Referring to Figure 6, let the values of these masses be: m_1 at each position of the nodal set $\{1, 4, 13, 16\}$; m_2 at each position of the nodal set $\{2, 3, 5, 8, 9, 12, 14, 15\}$; m_3 at each position of the nodal set $\{6, 7, 10, 11\}$.

The diagonal mass matrix $\mathbf{M}^{(i)}$ for a given subspace $U^{(i)}$ consists of non-zero diagonal elements m_{ii} $(i=1,2,\ldots,r)$, which are the values of the mass at each of the nodes of basis vector Φ_i. Thus, the mass matrices $\{\mathbf{M}^{(1)}, \mathbf{M}^{(2)}, \mathbf{M}^{(3)}, \mathbf{M}^{(4)}, \mathbf{M}^{(51)}, \mathbf{M}^{(52)}\}$ for subspaces $\{U^{(1)}, U^{(2)}, U^{(3)}, U^{(4)}, U^{(51)}, U^{(52)}\}$, respectively, are:

$$\mathbf{M}^{(1)} = \mathbf{M}^{(4)} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \tag{57}$$

$$\mathbf{M}^{(2)} = \mathbf{M}^{(3)} = [m_2] \tag{58}$$

$$\mathbf{M}^{(51)} = \mathbf{M}^{(52)} = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_3 & 0 & 0 \\ 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & m_2 \end{bmatrix} \tag{59}$$

4.3.4. Eigenvalues and eigenvectors. The system eigenvalues λ $(=1/\omega^2$, where ω is a natural circular frequency of the system) are obtained separately for each subspace, from the well-known condition

$$|\mathbf{A} - \lambda \mathbf{M}^{-1}| = 0 \tag{60}$$

in which the flexibility matrix $\mathbf{A}$ must now be taken as the subspace flexibility matrix, consisting of elements a_{ij} $(i=1,2,\ldots,r;\ j=1,2,\ldots,r)$; the mass matrix $\mathbf{M}$ must now be taken as the subspace mass matrix, consisting of non-zero diagonal elements m_{ii} $(i=1,2,\ldots,r)$. The eigenvalues yielded from the individual subspaces are the actual eigenvalues of the original problem.

Eigenvectors $\boldsymbol{\psi}$ are also obtained separately for each subspace, by substituting the now-known r eigenvalues of the subspace, one at a time, into the well-known relation

$$(\mathbf{A} - \lambda \mathbf{M}^{-1})\boldsymbol{\psi} = 0 \tag{61}$$

Writing this in expanded form for a given subspace, we obtain

$$\begin{bmatrix} (a_{11} - (\lambda/m_{11})) & a_{12} & \cdot & a_{1r} \\ a_{21} & (a_{22} - (\lambda/m_{22})) & \cdot & a_{2r} \\ \cdot & \cdot & \cdot & \cdot \\ a_{r1} & a_{r2} & \cdot & (a_{rr} - (\lambda/m_{rr})) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \cdot \\ \psi_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 0 \end{bmatrix} \tag{62}$$

If we substitute a particular eigenvalue of the subspace into Equation (62), and solve for the components $\psi_1, \psi_2, \ldots, \psi_r$, we will obtain the eigenvector $[\psi_1 \ \psi_2 \ \cdot \ \psi_r]^{\mathrm{T}}$ corresponding to that

DOI: 10.1002/nme

eigenvalue, which is an eigenvector in the *r-dimensional subspace*. Noting that the components $\psi_1, \psi_2, \ldots, \psi_r$ of such an eigenvector correspond to the basis vectors $\Phi_1, \Phi_2, \ldots, \Phi_r$, respectively, of the subspace in question, the eigenvector *in the original n-dimensional vector space of the problem* (associated with a particular natural frequency of the system) would then be given by allocating the calculated value of a subspace–eigenvector component to *all* the cable nodes associated with the corresponding basis vector, with the signs (positive or negative) of the allocations being in accordance with those of the basis-vector coordinates (Equations (41)–(47)).

In summary, we note that instead of solving an nth-degree polynomial characteristic equation in λ for the n roots of λ (i.e. eigenvalues), we needed only solve, independently of each other, a series of lower-degree characteristic equations in λ. For the example of the 16-node cable net, the decomposition led to the following mutually-independent characteristic equations in λ: two 3rd-degree, two 1st-degree, and one 4th-degree (the other 4th degree equation being identical to this).

An attribute of the group-theoretic approach is its ability to predict the shape of vibration modes, and of the number of modes of vibration of a given symmetry type. The occurrence of degenerate modes (that is, modes of the same frequency of vibration) is also predicted. The latter are always associated with irreducible representations of dimension greater than one. The number of linearly independent modes of a particular frequency is simply given by the dimension of the corresponding irreducible representation. For the C_{4v} symmetry group, the fifth irreducible representation $R^{(5)}$ is 2-dimensional; hence, the corresponding subspace $U^{(5)}$ is always associated with doubly repeating roots. Here we see again the ability of the group-theoretic approach to reveal important qualitative insights about the nature of the problem, before actual calculations for natural frequencies and exact mode shapes are performed.

5. STATIC ANALYSIS

Computational problems of the static analysis of indeterminate structures fall into one or the other of two broad categories: those based on a stiffness formulation of the generalized force-displacement relations (where generalized displacements are the unknowns to be solved for), and those based on a flexibility formulation (where generalized forces are the unknowns of the problem). Group-theoretic flexibility formulations for the analysis of symmetric plane and space frames have been given by Zlokovic [14] and by Zingoni *et al.* [45, 46]. Stiffness formulations have also been developed, particularly for the direct-stiffness method [14, 47]. Unlike eigenvalue problems, where the eigenvalues calculated within the independent subspaces of the problem are in fact the exact eigenvalues of the original problem, superposition of the obtained subspace results is necessary in problems of the static analysis of symmetric structures, in order to come up with the final effects.

It is well known that any non-symmetric arrangement of loading acting on a symmetric structure (for example, a beam or a portal frame) can be split into symmetric and anti-symmetric components, allowing advantage to be taken of the ensuing symmetry. Group theory offers the opportunity to extend this idea to more complex symmetric structures. As long as the structure itself (i.e. member and support configuration) is symmetric, the applied loading need not have this symmetry; group-theory can be used to decompose any arbitrary loading into symmetry-adapted components for allocation to the various subspaces of the problem [48].

Focussing their attention on the equilibrium matrix of a symmetric structure, Kangwai and Guest [49] employ group representation theory to cast this into block-diagonal form. Instead of

DOI: 10.1002/nme

using the idempotents of a group to generate the symmetry-adapted variables of the problem (as was done in the illustrative examples of the previous section), these investigators directly use the irreducible representations of the group. Such an approach has a particular advantage in dealing with subspaces associated with irreducible representations of dimension 2 (such as the third irreducible representation of the triangular group C_{3v}, or the fifth irreducible representation of the square group C_{4v}), since the subspace is decomposed directly into two smaller subspaces, without the additional considerations of the type that were necessary in Section 4.3.1. However, for problems involving only one-dimensional irreducible representations (e.g. problems of the rectangular group C_{2v}), the idempotent approach seems much simpler and quicker.

Kangwai and Guest are also able to extract other benefits from their group-theoretic formulation, such as simplification of the identification of states of self-stress and of mechanisms. Yet again we see the merits of group theory in providing insights into the mechanics of a problem.

Overall, the same type of group-theoretic simplifications noted for eigenvalue problems are also evident in the static analysis of structures, but for relatively small problems with low-order symmetry, the costs of decomposition and final superposition may not justify the approach. Compared with the areas of bifurcation and vibration analysis, research activity in applying group theory to the static analysis of structures has been rather limited, because the computational rewards are generally not as attractive.

6. FINITE ELEMENT ANALYSIS

6.1. Symmetry in finite elements

We have already seen some group-theoretic finite element formulations in the context of bifurcation analysis of cylindrical shells [28]. One of the earliest demonstrations of the applicability of group-theoretic methods to finite element analysis is due to Andersen and Noor [50], who used a group-theoretic approach jointly with symbolic integration for the finite element formulation for geometrically non-linear, anisotropic shallow shells, with effects of shear deformation taken into account. The dihedral symmetry groups D_4 (with the symmetry of the square) and D_3 (with the symmetry of the equilateral triangle) reduce the number of symbolic integrations (of shape-function quantities) required for quadrilateral and triangular elements, respectively.

In finite element problems, the variables of interest are the degrees of freedom associated with a given finite element, but what requires to be determined at the elemental level are usually the element matrices such as stiffness and mass matrices. The numerical integration associated with the evaluation of these matrices can involve the expenditure of huge computational effort, particularly in the case of new finite element proposals with large numbers of nodes and degrees of freedom. This is where a group-theoretic formulation can be most advantageous in reducing computational effort, when deliberate choice is made of symmetric finite elements.

As long as the shape of the element and the configuration of its nodes conform to a known symmetry type, group theory may be used to decompose the displacement field of the element, leading to symmetry-adapted shape functions within the independent subspaces, on the basis of which simpler integrations are performed to obtain subspace matrices. Straightforward additive combination of these subspace matrices, followed by a simple coordinate transformation, allows the conversion of the results to conventional form. Detailed features of a group-theoretic finite element formulation may be seen in the work of Zlokovic [51]. Elsewhere, displacement field

DOI: 10.1002/nme

decomposition has been illustrated, and consistent-mass matrices computed, for truss and beam finite elements [52], plate-bending elements [53] and rectangular solid elements [54].

Almost 10 years ago, Wohlever [55] noted that while the development of group-theoretic formulations for the bifurcation analysis of symmetric structures has progressed considerably, there is a lack of general finite element programmes that exploit group-theoretic concepts. In the formulation that he goes on to propose, he transforms the variables of the problem to a symmetry-adapted basis, enabling a decoupling of the incremental equations of the non-linear formulation, which can readily be solved. The procedure is applied to the buckling analysis of plates and shells of revolution. Mohan and Pratap [56] follow a similar procedure in their finite element analysis of the free vibration of shells with dihedral symmetry.

In both the approaches of Wohlever [55] and of Mohan and Pratap [56], the symmetry-adapted coordinates are used in only one stage of the finite element analysis (that is, in the solution of the incremental equations or solution for the eigenvalues), with the other stages of the process being standard. Thus, in modular finite element programmes, only this stage needs to be formulated in a group-theoretic manner, and incorporated within the whole.

Let us consider rectangular hexahedral elements with three degrees of freedom (u, v, w) per node (these being translations in the x, y, z directions, respectively), and having any number and arrangement of nodes consistent with the symmetry group D_{2h}. The symmetry group D_{2h} has eight symmetry elements: e (identity), c_2^z (rotation of π about the z axis), c_2^y (rotation of π about the y axis), c_2^x (rotation of π about the x axis), i (inversion through the centre of symmetry), σ_{xy} (reflection in the xy plane), σ_{xz} (reflection in the xz plane) and σ_{yz} (reflection in the yz plane). Typical elements conforming to this group are the 8-node 24 d.o.f. linear-displacement, 20-node 60 d.o.f. quadratic-displacement and 32-node 96 d.o.f. cubic-displacement brick elements.

6.2. Choice of origin, node numbering and positive directions

To achieve the best possible performance of the group-theoretic procedure in the computation of element properties, it is necessary to adopt a choice of origin, sequence of node numbering, and positive directions for freedoms that reflect the intrinsic symmetry of the element. The origin of the local coordinate system of the element should be chosen at the centre of symmetry of the configuration. Having done this, one may then proceed to assign a numbering for nodes and freedoms in accordance with the following procedure (details may be seen in Reference [57]).

Node 1 is chosen in the positive–positive–positive octant of the Cartesian coordinate system. Defining a nodal set as the collection of all the nodal positions of the element that are permuted or interchanged among themselves by symmetry operations of the associated symmetry group, the rest of the nodes belonging to the same nodal set as node 1 are numbered consecutively in the order generated by permuting node 1 through each symmetry operation of the group D_{2h}, such symmetry operations being performed in the same order as appears across the top of the character table of the symmetry group (that is, in the order e, c_2^z, c_2^y, c_2^x, i, σ_{xy}, σ_{xz} and σ_{yz}). This sequence of node numbering is illustrated for the 8-node 24 d.o.f. element in Figure 8(a). For elements with more than 8 nodes (e.g. 32 nodes), the procedure is repeated for all nodal sets of the element's nodal positions until all the n individual nodes of the element have been numbered consecutively from 1 to n.

Having numbered all the nodes of the finite element, the positive directions of the freedoms and loads at node 1 may be chosen arbitrarily, but having fixed these, the positive directions for the rest of the nodal freedoms and nodal loads of the element should be chosen such that the

DOI: 10.1002/nme

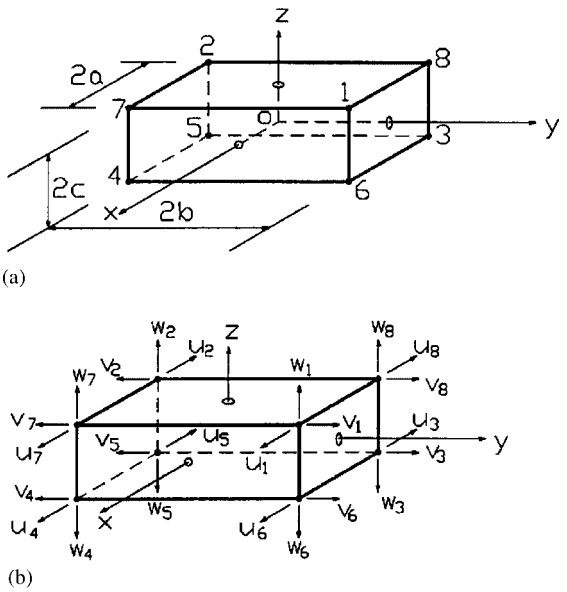

Figure 8. Symmetry-adapted conventions for the 8-node 24 d.o.f. rectangular solid element: (a) node numbering and (b) positive directions for nodal freedoms.

ensuing overall pattern of directions of nodal quantities preserves the full symmetry of the pattern of nodes. This convention is illustrated for the 8-node 24 d.o.f. element in Figure 8(b).

6.3. Symmetry-adapted shape functions

Let us consider a rectangular hexahedral element with n nodes (n being typically 8, 20, 32 or 64). The set $\{u_i, v_i, w_i\}$ of the three translational degrees of freedom at node i $(i=1,\ldots,n)$ may be denoted by $F_{(i)}$. The symmetry group D_{2h} is associated with eight idempotents $P^{(j)}$ $(j=1,2,\ldots,8)$ corresponding to its eight irreducible representations $R^{(j)}$ $(j=1,2,\ldots,8)$. These idempotents are as follows:

$$\begin{bmatrix} P^{(1)} \\ P^{(2)} \\ P^{(3)} \\ P^{(4)} \\ P^{(5)} \\ P^{(6)} \\ P^{(7)} \\ P^{(8)} \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} e \\ c_2^z \\ c_2^y \\ c_2^x \\ i \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} \tag{63}$$

Applying the first idempotent $P^{(1)}$ to each of the nodal-freedom sets $F_{(i)}$ $(i=1,\ldots,n)$ of the rectangular hexahedral element in question, we obtain n linear combinations of the $F_{(i)}$ (with up

DOI: 10.1002/nme

to eight different $F_{(i)}$ per linear combination), from which we can identify a set of $r^{(1)}$ ($r^{(1)}<n$) *independent* linear combinations, namely $\phi_1^{(1)}, \phi_2^{(1)}, \ldots, \phi_{r^{(1)}}^{(1)}$, as the basis vectors or symmetry-adapted nodal-freedom sets for subspace $U^{(1)}$. Similarly, the second idempotent $P^{(2)}$ yields $r^{(2)}$ symmetry-adapted nodal-freedom sets for subspace $U^{(2)}$, namely $\phi_1^{(2)}, \phi_2^{(2)}, \ldots, \phi_{r^{(2)}}^{(2)}$, and so on. The $r^{(j)}$ ($j=1,2,\ldots,8$) symmetry-adapted nodal-freedom sets for subspace $U^{(j)}$, namely $\phi_1^{(j)}, \phi_2^{(j)}, \ldots, \phi_{r^{(j)}}^{(j)}$, may be collected together as the column vector $\mathbf{\Phi}^{(j)}$. Thus, the dimensions of subspaces $U^{(1)}, U^{(2)}, \ldots, U^{(8)}$ are $r^{(1)}, r^{(2)}, \ldots, r^{(8)}$, respectively, and $r^{(1)}+r^{(2)}+\cdots+r^{(8)}=n$. For the 8-node 24 d.o.f. element, we obtain (noting that each subspace for this very simple element is one-dimensional, that is, $r^{(j)}=1$ for $j=1,2,\ldots,8$):

$$\begin{bmatrix} \mathbf{\Phi}^{(1)} \\ \mathbf{\Phi}^{(2)} \\ \mathbf{\Phi}^{(3)} \\ \mathbf{\Phi}^{(4)} \\ \mathbf{\Phi}^{(5)} \\ \mathbf{\Phi}^{(6)} \\ \mathbf{\Phi}^{(7)} \\ \mathbf{\Phi}^{(8)} \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} F_{(1)} \\ F_{(2)} \\ F_{(3)} \\ F_{(4)} \\ F_{(5)} \\ F_{(6)} \\ F_{(7)} \\ F_{(8)} \end{bmatrix} \tag{64}$$

The assumed displacement polynomial for u, v, w is usually of the same form. Therefore, we need only consider one displacement component, say u, and allocate the terms of the displacement polynomial to the subspaces of the symmetry group D_{2h}, in accordance with the symmetry types of the displacement fields associated with these terms. The resulting eight internal-displacement allocations $u^{(1)}, u^{(2)}, \ldots, u^{(8)}$, corresponding to subspaces $U^{(1)}, U^{(2)}, \ldots, U^{(8)}$, respectively, are functions of the coordinate variables x, y, z and the polynomial coefficients (say $\alpha_1, \alpha_2, \ldots, \alpha_\eta$) of the problem. For the 8-node 24 d.o.f. element, if we adopt the displacement polynomial:

$$u(x,y,z)=\alpha_1+\alpha_2 x+\alpha_3 y+\alpha_4 z+\alpha_5 xy+\alpha_6 yz+\alpha_7 xz+\alpha_8 xyz \tag{65}$$

then the subspace allocations of terms of the polynomial are as follows:

$$\begin{bmatrix} u^{(1)}(x,y,z) \\ u^{(2)}(x,y,z) \\ u^{(3)}(x,y,z) \\ u^{(4)}(x,y,z) \\ u^{(5)}(x,y,z) \\ u^{(6)}(x,y,z) \\ u^{(7)}(x,y,z) \\ u^{(8)}(x,y,z) \end{bmatrix} = \begin{bmatrix} x & & & & & & & \\ & y & & & & & & \\ & & z & & & & & \\ & & & xyz & & & & \\ & & & & yz & & & \\ & & & & & xz & & \\ & & & & & & xy & \\ & & & & & & & 1 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_8 \\ \alpha_6 \\ \alpha_7 \\ \alpha_5 \\ \alpha_1 \end{bmatrix} \tag{66}$$

DOI: 10.1002/nme

If we substitute the nodal coordinates of the first nodes of the nodal sets of the element into the decomposed polynomial, we automatically obtain the sets $\mathbf{\Phi}_u^{(j)}$ of the symmetry-adapted nodal displacements for subspace $U^{(j)}$, in terms of the polynomial coefficients. Rearrangement then yields the polynomial coefficients in terms of the symmetry-adapted nodal displacements $\mathbf{\Phi}_u^{(j)}$, enabling the symmetry-adapted internal displacements $u^{(j)}$ to be written down in terms of the symmetry-adapted nodal displacements $\mathbf{\Phi}_u^{(j)}$:

$$u^{(j)} = \mathbf{S}^{(j)}(x, y, z)\mathbf{\Phi}_u^{(j)} \tag{67}$$

By definition, $\mathbf{S}^{(j)}(x, y, z)$ must be the symmetry-adapted shape functions for subspace $U^{(j)}$. The $\mathbf{S}^{(j)}(x, y, z)$ for the 8-node 24 d.o.f. element happen to be 1×1 matrices:

$$\mathbf{S}^{(1)} = \frac{x}{a} \tag{68a}$$

$$\mathbf{S}^{(2)} = \frac{y}{b} \tag{68b}$$

$$\mathbf{S}^{(3)} = \frac{z}{c} \tag{68c}$$

$$\mathbf{S}^{(4)} = \frac{xyz}{abc} \tag{68d}$$

$$\mathbf{S}^{(5)} = \frac{yz}{bc} \tag{68e}$$

$$\mathbf{S}^{(6)} = \frac{xz}{ac} \tag{68f}$$

$$\mathbf{S}^{(7)} = \frac{xy}{ab} \tag{68g}$$

$$\mathbf{S}^{(8)} = 1 \tag{68h}$$

6.4. Symmetry-adapted element matrices

Similar to the procedure followed in conventional considerations, we obtain the symmetry-adapted element matrices for subspace $U^{(j)}$ by integrating the associated symmetry-adapted shape functions, but owing to the decomposition feature inherent in the group-theoretic approach, this integration is performed over only the positive–positive–positive octant of the element. For instance, the symmetry-adapted element stiffness and consistent-mass matrices, $\mathbf{K}^{(j)}$ and $\mathbf{M}^{(j)}$ for subspace $U^{(j)}$, are obtained as

$$\mathbf{K}^{(j)} = \int_V [\mathbf{B}^{(j)}]^{\mathrm{T}} \mathbf{\Omega} \mathbf{B}^{(j)} \, \mathrm{d}V \tag{69a}$$

$$\mathbf{M}^{(j)} = \int_V \rho [\mathbf{S}^{(j)}]^{\mathrm{T}} \mathbf{S}^{(j)} \, \mathrm{d}V \tag{69b}$$

where $\mathbf{B}^{(j)}$ and $\mathbf{S}^{(j)}$ are the subspace strain–displacement and shape-function matrices, $\mathbf{\Omega}$ is a matrix of material stiffnesses and ρ is the material mass density. For the 8-node 24 d.o.f. element,

DOI: 10.1002/nme

the consistent-mass matrices for the eight subspaces are obtained as follows:

$$\mathbf{M}^{(1)} = \rho \int_0^c \int_0^b \int_0^a \mathbf{S}^{(1)}\mathbf{S}^{(1)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{\rho abc}{3} \tag{70a}$$

$$\mathbf{M}^{(2)} = \rho \int_0^c \int_0^b \int_0^a \mathbf{S}^{(2)}\mathbf{S}^{(2)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{\rho abc}{3} \tag{70b}$$

$$\mathbf{M}^{(3)} = \rho \int_0^c \int_0^b \int_0^a \mathbf{S}^{(3)}\mathbf{S}^{(3)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{\rho abc}{3} \tag{70c}$$

$$\mathbf{M}^{(4)} = \rho \int_0^c \int_0^b \int_0^a \mathbf{S}^{(4)}\mathbf{S}^{(4)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{\rho abc}{27} \tag{70d}$$

$$\mathbf{M}^{(5)} = \rho \int_0^c \int_0^b \int_0^a \mathbf{S}^{(5)}\mathbf{S}^{(5)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{\rho abc}{9} \tag{70e}$$

$$\mathbf{M}^{(6)} = \rho \int_0^c \int_0^b \int_0^a \mathbf{S}^{(6)}\mathbf{S}^{(6)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{\rho abc}{9} \tag{70f}$$

$$\mathbf{M}^{(7)} = \rho \int_0^c \int_0^b \int_0^a \mathbf{S}^{(7)}\mathbf{S}^{(7)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{\rho abc}{9} \tag{70g}$$

$$\mathbf{M}^{(8)} = \rho \int_0^c \int_0^b \int_0^a \mathbf{S}^{(8)}\mathbf{S}^{(8)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho abc \tag{70h}$$

Notice how astonishingly simple the above results are. The subspace matrices are all single terms (i.e. 1×1 matrices). The conventional mass matrix for this element is 8×8 in size. The above group-theoretic results can be converted to conventional form through a superposition of the subspace matrices, and simple coordinate transformations [51–54].

6.5. Some numerical studies

The following example is taken from Reference [57]. Consider a 64-node 192 d.o.f solid hexahedral element with an arrangement of nodes conforming to the symmetry of the group D_{2h}. A 64-term polynomial may be assumed for the displacement field, as follows:

$$\begin{aligned} u(x,y,z) = {} & \alpha_1+\alpha_2x+\alpha_3y+\alpha_4z+\alpha_5x^2+\alpha_6xy+\alpha_7xz+\alpha_8y^2+\alpha_9yz+\alpha_{10}z^2+\alpha_{11}x^3+\alpha_{12}x^2y \\ & +\alpha_{13}x^2z+\alpha_{14}xy^2+\alpha_{15}xyz+\alpha_{16}y^3+\alpha_{17}y^2z+\alpha_{18}z^2x+\alpha_{19}z^2y+\alpha_{20}z^3 \\ & +\alpha_{21}x^3y+\alpha_{22}x^3z+\alpha_{23}x^2y^2+\alpha_{24}x^2yz+\alpha_{25}x^2z^2+\alpha_{26}xy^3+\alpha_{27}xy^2z \\ & +\alpha_{28}xyz^2+\alpha_{29}xz^3+\alpha_{30}y^3z+\alpha_{31}y^2z^2+\alpha_{32}yz^3+\alpha_{33}x^3y^2 \\ & +\alpha_{34}x^3yz+\alpha_{35}x^3z^2+\alpha_{36}x^2y^3+\alpha_{37}x^2y^2z+\alpha_{38}x^2yz^2+\alpha_{39}x^2z^3 \\ & +\alpha_{40}xy^3z+\alpha_{41}xy^2z^2+\alpha_{42}xyz^3+\alpha_{43}y^3z^2+\alpha_{44}y^2z^3+\alpha_{45}x^3y^3+\alpha_{46}x^3y^2z \\ & +\alpha_{47}x^3yz^2+\alpha_{48}x^3z^3+\alpha_{49}x^2y^3z+\alpha_{50}x^2y^2z^2+\alpha_{51}x^2yz^3+\alpha_{52}xy^3z^2 \\ & +\alpha_{53}xy^2z^3+\alpha_{54}y^3z^3+\alpha_{55}x^3y^3z+\alpha_{56}x^3y^2z^2+\alpha_{57}x^3yz^3+\alpha_{58}x^2y^3z^2 \\ & +\alpha_{59}x^2y^2z^3+\alpha_{60}xy^3z^3+\alpha_{61}x^3y^3z^2+\alpha_{62}x^3y^2z^3+\alpha_{63}x^2y^3z^3+\alpha_{64}x^3y^3z^3 \end{aligned} \tag{71}$$

DOI: 10.1002/nme

We may allocate the terms of the above displacement field to the respective subspaces $U^{(j)}$ $(j=1,2,\ldots,8)$ associated with the symmetry group D_{2h}, to obtain

$$u^{(1)}(x,y,z)=\alpha_1+\alpha_5x^2+\alpha_8y^2+\alpha_{10}z^2+\alpha_{23}x^2y^2+\alpha_{25}x^2z^2+\alpha_{31}y^2z^2+\alpha_{50}x^2y^2z^2 \quad (72a)$$

$$u^{(2)}(x,y,z)=\alpha_6xy+\alpha_{21}x^3y+\alpha_{26}xy^3+\alpha_{28}xyz^2+\alpha_{45}x^3y^3+\alpha_{47}x^3yz^2 +\alpha_{52}xy^3z^2+\alpha_{61}x^3y^3z^2 \quad (72b)$$

$$u^{(3)}(x,y,z)=\alpha_7xz+\alpha_{22}x^3z+\alpha_{27}xy^2z+\alpha_{29}xz^3+\alpha_{46}x^3y^2z +\alpha_{48}x^3z^3+\alpha_{53}xy^2z^3+\alpha_{62}x^3y^2z^3 \quad (72c)$$

$$u^{(4)}(x,y,z)=\alpha_9yz+\alpha_{24}x^2yz+\alpha_{30}y^3z+\alpha_{32}yz^3+\alpha_{49}x^2y^3z +\alpha_{51}x^2yz^3+\alpha_{54}y^3z^3+\alpha_{63}x^2y^3z^3 \quad (72d)$$

$$u^{(5)}(x,y,z)=\alpha_{15}xyz+\alpha_{34}x^3yz+\alpha_{40}xy^3z+\alpha_{42}xyz^3+\alpha_{55}x^3y^3z +\alpha_{57}x^3yz^3+\alpha_{60}xy^3z^3+\alpha_{64}x^3y^3z^3 \quad (72e)$$

$$u^{(6)}(x,y,z)=\alpha_4z+\alpha_{13}x^2z+\alpha_{17}y^2z+\alpha_{20}z^3+\alpha_{37}x^2y^2z +\alpha_{39}x^2z^3+\alpha_{44}y^2z^3+\alpha_{59}x^2y^2z^3 \quad (72f)$$

$$u^{(7)}(x,y,z)=\alpha_3y+\alpha_{12}x^2y+\alpha_{16}y^3+\alpha_{19}yz^2+\alpha_{36}x^2y^3 +\alpha_{38}x^2yz^2+\alpha_{43}y^3z^2+\alpha_{58}x^2y^3z^2 \quad (72g)$$

$$u^{(8)}(x,y,z)=\alpha_2x+\alpha_{11}x^3+\alpha_{14}xy^2+\alpha_{18}xz^2+\alpha_{33}x^3y^2+\alpha_{35}x^3z^2 +\alpha_{41}xy^2z^2+\alpha_{56}x^3y^2z^2 \quad (72h)$$

Proceeding as in the previous example, we can generate any required element matrices. Taking the computation of the element mass matrix, for instance, we note that the conventional method requires the computation of 64 constants $\{\alpha_1,\alpha_2,\ldots,\alpha_{64}\}$ first, followed by the integration of 2080 distinct terms N_iN_j $(i=1,2,\ldots,64;\ j=1,2,\ldots,64;\ m_{ij}=m_{ji}$ for $i\neq j)$. On the other hand, the group-theoretic decomposition requires the computation of only eight constants for each subspace, followed by the integration of 36 distinct terms N_iN_j $(i=1,2,\ldots,8;\ j=1,2,\ldots,8;\ m_{ij}=m_{ji}$ for $i\neq j)$ for each subspace, which amount to 288 terms for the eight subspaces. A numerical study of the 64-node rectangular hexahedral element showed that the group-theoretic method is 11 times faster than the conventional approach in generating the elements of the mass matrix [57].

The extension of the group-theoretic finite element formulation to non-symmetric isoparametric shapes (the quadrilateral and hexahedron) has been demonstrated by Zlokovic and co-workers [58]. In a numerical study of the 4-noded quadrilateral element and the 8-noded hexahedral element, they show that computations based on the group-theoretic formulation are 4 times faster for quadrilateral elements and 2.5 times faster for hexahedral elements, in comparison with computations based on the conventional formulation. This more generalized formulation represents a significant step forward, as it opens the way to possible wider application of group theory in finite element computations. It allows the benefits of symmetry (i.e. group-theoretic simplifications) to be realized

DOI: 10.1002/nme

where no physical symmetry is actually present, provided an arbitrary non-symmetric element geometry can be linearly mapped onto a symmetric one.

Overall, group-theoretic formulations for finite elements result in considerable reductions in computational effort in comparison with conventional computations. However, it would appear that the formulations only become really advantageous in the case of finite elements with a high degree of symmetry and a large number of nodes and nodal degrees of freedom.

In a recent paper, Kaveh and Fazli [59] have combined group-theoretic techniques and graph theory to factorize the global mass and stiffness matrices of symmetric finite element models, thereby achieving considerable simplification in the analysis of the dynamic problem. Here, the implementation of the group-theoretic simplification is at the level of the assembled structure, rather than element level. While a considerable amount of research has already been reported on group-theoretic formulations of computations at the element level, relatively less has been reported on FEM formulations at the structure level, thus, this line of work is a good development. A fully group-theoretic FEM procedure would be one that employs group-theoretic formulations both at the element level (computations of element matrices) and at the structure level (assembly of global matrices), with final output quantities (displacements, stresses, eigenvalues, modes, etc.) available in conventional form.

7. SUMMARY AND CONCLUSIONS

While physicists and chemists have long recognized and exploited group theory in the analysis of various problems involving symmetry, significant efforts towards application of the same ideas to engineering problems only started in the past 40 years or so, with problems of bifurcation, vibration and finite element modelling having received the most attention.

This paper has reviewed applications of group theory in solid and structural mechanics. The key features of the approach in the contexts of these different problems have been identified, and illustrated by reference to some recent studies of the author. In all cases, group theory has been shown to be highly effective and systematic in exploiting the symmetry of a structural system, leading to simplifications in the analysis and considerable reductions in computational effort.

The next line of research in this area could be the extension of group-theoretic formulations to symmetry-deficient structures (corrections for ‘lack of symmetry’), which would open up vast opportunities for application of group theory, as many structures belong to this category. Some investigators have already taken the lead in this direction [13]. Also, for group-theoretic formulations to enter into mainstream engineering calculations, it is clear that effective and efficient computer algorithms need to be developed, to handle steps like:

- automatic recognition of symmetry in a structure
- identification of the most appropriate symmetry group for the problem
- creation of the relevant subspaces for the decomposed problem
- generation of symmetry-adapted variables within these subspaces
- identification of the basis vectors for the various subspaces
- calculations of the relevant quantities of interest within the independent subspaces
- superposition of subspace results to give quantities of interest in conventional form.

DOI: 10.1002/nme

On the FEM front, an integrated group-theoretic procedure is required, capable of taking into account the existence of symmetry (and correcting for the lack of it) at both the element and the structure level. The observation on the lack of general group-theoretic finite element programmes, noted by Wohlever [55] almost a decade ago, is still as true today as it was then. For large-scale numerical problems, the decomposition feature of the group-theoretic approach allows the possible use of parallel processors in computations, which would result in further reductions in computational time.

8. FINAL NOTE

The author previously reviewed this subject 7 years ago [60]. Since then, some new developments have occurred in the field, necessitating an expanded treatment of the subject, and a re-appraisal of the field in the light of these new developments. This is a steadily growing area, and the next review of the subject will probably reflect significant progress on some of the aspects identified for future research in the last section.

REFERENCES

1. Glockner PG. Symmetry in structural mechanics. *ASCE Journal of the Structural Division* 1973; **99**(ST1):71–89.
2. Weyl H. *The Theory of Groups and Quantum Mechanics*. Dover Publications: New York, 1932.
3. Wigner EP. *Group Theory and its Applications to the Quantum Mechanics of Atomic Spectra*. Academic Press: New York, 1959.
4. Hamermesh M. *Group Theory and its Application to Physical Problems*. Pergamon Press: Oxford, 1962.
5. Schonland D. *Molecular Symmetry*. Van Nostrand: London, 1965.
6. Loebl EM (ed.). *Group Theory and its Applications*, vols 1 and 2. Academic Press: New York, 1969, 1971.
7. Wherrett BS. *Group Theory for Atoms*, *Molecules and Solids*. Prentice-Hall: Englewood Cliffs, NJ, 1986.
8. Chou TT, Yang CN. Exact solution of the vibration problem for the carbon-60 molecule. *Physics Letters A* 1997; **235**:97–104.
9. Kardestuncer H, Berg K. Matrix analysis of large symmetric skeletal systems. In *Proceedings of the Conference on Symmetry*, *Similarity and Group Theoretic Methods in Mechanics*, Glockner PG, Singh MC (eds), Calgary, 1974; 623–632.
10. Hussey MJL. General theory of cyclically symmetric frames. *ASCE Journal of the Structural Division* 1967; **93**(ST2):163–176.
11. Thomas DL. Dynamics of rotationally periodic structures. *International Journal for Numerical Methods in Engineering* 1979; **14**:81–102.
12. Ramamurti V. Comparison of semi analytic and cyclic symmetric approaches. *Computers and Structures* 1995; **54**(6):1105–1111.
13. Varkonyi PL, Domokos G. Imperfect symmetry: a new approach to structural optima via group representation theory. *International Journal of Solids and Structures* 2007; **44**(14,15):4723–4741.
14. Zlokovic GM. *Group Theory and G-Vector Spaces in Structural Analysis*. Ellis Horwood: Chichester, U.K., 1989.
15. Olver PJ. *Applications of Lie Groups to Differential Equations*. Springer: Berlin, 1986.
16. Ovsiannikov LV. *Group Analysis of Differential Equations*. Academic Press: New York, 1982.
17. Djondjorov PA. Invariant properties of Timoshenko beam equations. *International Journal of Engineering Science* 1995; **33**(14):2103–2114.
18. Vassilev VM. Application of Lie groups to the theory of shells and rods. *Proceedings of the Second World Congress of Nonlinear Analysts*: *Nonlinear Analysis*, *Theory*, *Methods and Applications*, Athens, Greece, vol. 30(8), 1997; 4839–4848.
19. Nardinocchi P, Teresi L, Tiero A. A direct theory of affine rods. *European Journal of Mechanics*: *A/Solids* 2002; **21**(4):653–667.
20. Djondjorov PA, Vassilev VM. Conservation laws and group-invariant solutions of the von Kármán equations. *International Journal for Nonlinear Mechanics* 1996; **31**(1):73–87.

DOI: 10.1002/nme

21. Renton JD. On the stability analysis of symmetrical frameworks. *Quarterly Journal of Mechanics and Applied Mathematics* 1964; **17**:175–197.
22. Sattinger DH. *Group Theoretic Methods in Bifurcation Theory*. Lecture Notes in Mathematics, vol. 762. Springer: Berlin, 1979.
23. Golubitsky M, Schaeffer DG. *Singularities and Groups in Bifurcation Theory*, vol. 1. Springer: Berlin, 1985.
24. Golubitsky M, Schaeffer DG. *Singularities and Groups in Bifurcation Theory*, vol. 2. Springer: Berlin, 1988.
25. Buzano E. Secondary bifurcations of a thin rod under axial compression. *SIAM Journal on Mathematical Analysis* 1986; **17**(2):312–321.
26. Ikeda K, Murota K, Fujii H. Bifurcation hierarchy of symmetric structures. *International Journal of Solids and Structures* 1991; **27**(12):1551–1573.
27. Ikeda K, Murota K. Bifurcation analysis of symmetric structures using block-diagonalisation. *Computer Methods in Applied Mechanics and Engineering* 1991; **86**:215–243.
28. Wohlever JC, Healey TJ. A group-theoretic approach to the global bifurcation analysis of an axially compressed cylindrical shell. *Computer Methods in Applied Mechanics and Engineering* 1995; **122**:315–349.
29. Healey TJ. A group-theoretic approach to computational bifurcation problems with symmetry. *Computer Methods in Applied Mechanics and Engineering* 1988; **67**:257–295.
30. Ikeda K, Murota K, Yanagimoto A, Noguchi H. Improvement of the scaled corrector method for bifurcation analysis using symmetry-exploiting block-diagonalization. *Computer Methods in Applied Mechanics and Engineering* 2007; **196**:1648–1661.
31. Ikeda K, Nakazawa M. Bifurcation hierarchy of a rectangular plate. *International Journal of Solids and Structures* 1998; **35**(7/8):593–617.
32. Aston PJ. Analysis and computation of symmetry-breaking bifurcation and scaling laws using group-theoretic methods. *SIAM Journal on Mathematical Analysis* 1991; **22**(1):181–212.
33. Ikeda K, Murota K. Imperfect bifurcation in structures and materials. *Engineering Use of Group-Theoretic Bifurcation Theory*. Springer: Berlin, 2002.
34. Yang TL. Symmetry properties and normal modes of vibration. *Journal of Nonlinear Mechanics* 1968; **3**:367–381.
35. Singh MC, Mishra AK. Analysis of modes of vibrations of symmetric networks by the group representation theory. In *Proceedings of the Third Canadian Congress of Applied Mechanics*, Glockner PG (ed.), Calgary, 1971; 127–128.
36. Healey TJ, Treacy JA. Exact block diagonalisation of large eigenvalue problems for structures with symmetry. *International Journal for Numerical Methods in Engineering* 1991; **31**:265–285.
37. Zingoni A. An efficient computational scheme for the vibration analysis of high-tension cable nets. *Journal of Sound and Vibration* 1996; **189**(1):55–79.
38. Zingoni A, Pavlovic MN. On natural-frequency determination of symmetric grid-mass systems. In *Structural Dynamics*: *Recent Advances*, Ferguson NS, Wolfe HF, Mei C (eds). Institute of Sound and Vibration Research: Southampton, 1994; 151–163.
39. Zingoni A, Pavlovic MN, Lloyd-Smith D, Zlokovic GM. Group-theory considerations of finite-difference plate eigenvalue problems. In *Developments in Computational Engineering Mechanics*, Topping BHV (ed.). Civil-Comp Press: Edinburgh, 1993; 243–256.
40. Zingoni A. On the symmetries and vibration modes of layered space grids. *Engineering Structures* 2005; **27**(4):629–638.
41. Mohan SJ, Pratap R. A natural classification of vibration modes of polygonal ducts based on group theoretic analysis. *Journal of Sound and Vibration* 2004; **269**(3–5):745–764.
42. Kaveh A, Sayarinejad MA. Graph symmetry in dynamic systems. *Computers and Structures* 2004; **82**:2229–2240.
43. Kaveh A, Nikbakht M. Decomposition of symmetric mass–spring vibrating systems using groups, graphs and linear algebra. *Communications in Numerical Methods in Engineering* 2007; **23**(7):639–664.
44. Zingoni A. On group-theoretic computation of natural frequencies for spring–mass dynamic systems with rectilinear motion. *Communications in Numerical Methods in Engineering* 2008; **24**:973–987.
45. Zingoni A, Pavlovic MN, Zlokovic GM. A symmetry-adapted flexibility approach for multi-storey space frames: general outline and symmetry-adapted redundants. *Structural Engineering Review* 1995; **7**(2):107–119.
46. Zingoni A, Pavlovic MN, Lloyd-Smith D, Zlokovic GM. Application of group theory to the analysis of space frames. In *Space Structures*, Parke GAR, Howard CM (eds). Thomas Telford: London, 1993; 1334–1347.
47. Zingoni A, Pavlovic MN, Zlokovic GM. Symmetry and the direct stiffness method in structural analysis: a formulation based on group theory. In *Advances in Computational Mechanics*, Papadrakakis M, Topping BHV (eds). Civil-Comp Press: Edinburgh, 1994; 107–115.

DOI: 10.1002/nme

48. Zingoni A, Pavlovic MN, Zlokovic GM. A symmetry-adapted flexibility approach for multi-storey space frames: symmetry-adapted loads. *Structural Engineering Review* 1995; **7**(2):121–130.
49. Kangwai RD, Guest SD. Symmetry-adapted equilibrium matrices. *International Journal of Solids and Structures* 2000; **37**(11):1525–1548.
50. Andersen CM, Noor AK. Use of group-theoretic methods in the development of nonlinear shell finite elements. In *Proceedings of the Conference on Symmetry, Similarity and Group Theoretic Methods in Mechanics*, Glockner PG, Singh MC (eds), Calgary, 1974; 533–558.
51. Zlokovic GM. *Group Supermatrices in Finite Element Analysis*. Ellis Horwood: Chichester, U.K., 1992.
52. Zingoni A. Truss and beam finite elements revisited: a derivation based on displacement-field decomposition. *International Journal of Space Structures* 1996; **11**(4):371–380.
53. Zingoni A. Symmetry-adapted shape functions for rectangular plate-bending elements. In *Proceedings of the Sixth International Conference on Steel and Space Structures*, Chiew SP (ed.). Singapore Structural Steel Society: Singapore, 1999; 133–148.
54. Zingoni A. Group-theoretic computation of matrices for rectangular hexahedral finite elements. In *Proceedings of the First MIT Conference on Computational Fluid and Solid Mechanics*, Bathe KJ (ed.). Elsevier Science: Oxford, 2001; 1683–1685.
55. Wohlever JC. Some computational aspects of a group theoretic finite element approach to the buckling and postbuckling analyses of plates and shells-of-revolution. *Computer Methods in Applied Mechanics and Engineering* 1999; **170**:373–406.
56. Mohan SJ, Pratap R. A group theoretic approach to the linear free vibration analysis of shells with dihedral symmetry. *Journal of Sound and Vibration* 2002; **252**(2):317–341.
57. Zingoni A. A group-theoretic formulation for symmetric finite elements. *Finite Elements in Analysis and Design* 2005; **41**(6):615–635.
58. Zlokovic G, Maneski T, Nestorovic M. Group theoretical formulation of quadrilateral and hexahedral isoparametric finite elements. *Computers and Structures* 2004; **82**(11,12):883–899.
59. Kaveh A, Fazli H. Graph coloration and group theory in dynamic analysis of symmetric finite element models. *Finite Elements in Analysis and Design* 2007; **43**(11,12):901–911.
60. Zingoni A. Group-theoretic applications in solid and structural mechanics: a review. In *Computational Structures Technology*, Topping BHV, Bittnar Z (eds). Saxe-Coburg Publications: Stirling, Scotland, 2002; 283–317.

DOI: 10.1002/nme