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On group-theoretic computation of natural frequencies for spring–mass dynamic systems with rectilinear motion

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SUMMARY

In this article, group theory is employed to simplify the computation of natural circular frequencies for spring–mass dynamic systems with rectilinear motion. The systems are by themselves not physically symmetric (or they exhibit only very weak symmetry properties), but they can be transformed into graphs that preserve all the connectivities between masses and springs, while featuring the maximum possible symmetry. For physical dynamic systems exhibiting symmetry, a well-known group-theoretic approach involves the computation of the full stiffness matrix of the system first, followed by transformation of this matrix in order to cast it into block-diagonal form. The present approach involves the direct assembly of much smaller stiffness matrices within the decomposed subspaces of the problem, and is therefore computationally more efficient. Of particular focus in this study are transformed configurations belonging to 'triangular' symmetry groups, whose symmetries are difficult to exploit using conventional methods. It is shown how the repeating eigenvalues associated with the degenerate subspaces of such symmetry groups can easily be obtained. Copyright © 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Group theory can be used to simplify the analysis of a physical system, if such a system possesses symmetry properties [1]. In physics and chemistry, applications have ranged from problems in quantum mechanics [2, 3], to those involving molecular symmetry and crystallography [4, 5]. Recent years have seen innovative efforts to exploit similar ideas within the engineering sciences,

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or more specifically, in computational solid and structural mechanics [6–9]. Progress over the past 30 years may be seen in a recent review of the literature [10].

The basic idea is that if a physical system possesses symmetry properties which can be described by a group, the vector space V of the system can be decomposed, on the basis of representation theory of symmetry groups, into a number of independent subspaces $S^{(i)}$ $(i=1,2,\ldots,k)$ each of dimension a fraction of that of the original vector space. The required solutions for the system may then be obtained more easily by consideration of each of these subspaces separately, since they are independent. Typically, the matrix of equations describing the behaviour of the system ends up in block-diagonal form, allowing separation into independent sets of equations each corresponding to a subspace $S^{(i)}$ of V. In eigenvalue problems, it then becomes possible to generate all the eigenvalues of the original system by separately solving k smaller problems. This decomposition is the key feature of the group-theoretic formulation.

In vibration analysis, while block-diagonalization can be achieved by first assembling the full stiffness and mass matrix of the entire dynamic system, and then applying a group-theoretic transformation matrix upon these matrices to transform them into the required block-diagonal form [11], the block submatrices can be generated directly, as demonstrated in the work of Zlokovic [7], and the studies of Zingoni on the vibration of high-tension cable nets [12] and layered space grids [13].

Recently, Kaveh and Sayarinejad [14] have presented techniques for simplifying the calculation of eigenvalues of matrices with certain special patterns, such matrices representing symmetric graph models of various structures, including spring–mass systems [15]. In their most recent considerations of vibration problems of spring–mass systems that can be represented by symmetric graphs, Kaveh and Nikbakht [16] successfully bring together techniques of graphs and group theory, and employ the group-theoretic approach to simplify the calculation of eigenvalues of such systems. In their procedure, the full stiffness matrix of the entire system is first assembled, and then block-diagonalized *via* a transformation matrix.

In this article, spring–mass models of the type considered by Kaveh and Nikbakht [16] are revisited. The physical spring–mass systems are themselves not symmetric, or they exhibit only very minimal symmetry properties. They are first transformed into equivalent models that preserve all the graphical connectivities between masses and springs, while featuring the maximum possible symmetry. Group theory is then employed to simplify the extraction of eigenvalues for the equivalent system. The main difference between the procedure illustrated in this paper and the approach of Kaveh and Nikbakht is that here, the eigenvalue problem for each subspace is formulated directly, without the need for first assembling the full stiffness matrix for the entire system first. From a computational point of view, the present procedure therefore has improved efficiency, and complements the contribution of Kaveh and Nikbakht.

The symmetry properties of configurations belonging to the 'triangular' symmetry groups C_{3v} and C_{6v} are generally more difficult to exploit in comparison with those belonging to the 'rectangular' symmetry groups C_{2v} and C_{4v}. A secondary aim of this article is therefore to show how the repeating eigenvalues of the degenerate subspaces of the 'triangular' symmetry groups can be obtained in practice.

The fundamental concepts of symmetry groups and associated representation theory, as well as the key features of group-theoretic formulations, have already been described in the previous work of the author [9, 10, 12], and will not be repeated here. We will therefore go straight into the consideration of spring–mass systems of present interest, and through these numerical examples, illustrate the proposed computational procedure, and the gains that can be achieved.

2. A SIMPLE MODEL WITH C_{1v} SYMMETRY

To better understand the group-theoretic formulation, let us first consider a very simple example, namely the spring–mass system shown in Figure 1(a), where the degrees of freedom (d.o.f.s) are u_1 for mass 1 (m_1) and u_2 for mass 2 (m_2), both being positive when towards the right (as shown). For this problem, the two masses have the same value, that is, $m_1 = m_2 = m$. With the spring stiffnesses as shown, and considering small oscillations of the masses in the u direction, conventional considerations give the stiffness matrix **K** and mass matrix **M** as

$$\mathbf{K} = \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & (k_1+k_2) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \tag{1}$$

Ignoring damping, the eigenvalue condition

$$|\mathbf{K} - \omega^2 \mathbf{M}| = 0 \tag{2}$$

(where ω is a natural circular frequency of the system) gives

$$\begin{vmatrix} (k_1+k_2) - \omega^2 m & -k_2 \\ -k_2 & (k_1+k_2) - \omega^2 m \end{vmatrix} = 0 \tag{3a}$$

leading to the quadratic characteristic equation

$$[(k_1+k_2) - \omega^2 m]^2 - k_2^2 = 0 \tag{3b}$$

with roots

$$\omega_1^2 = \frac{k_1}{m}, \quad \omega_2^2 = \frac{k_1 + 2k_2}{m} \tag{4}$$

Clearly, this example is so trivial that we do not need to resort to group theory in order to simplify the computation of the circular frequencies. However, to bring out the essence of the proposed group-theoretic approach, we will re-consider the same problem. In Figure 1(b) is shown a spring–mass system that is equivalent to the original spring–mass system of Figure 1(a). All the

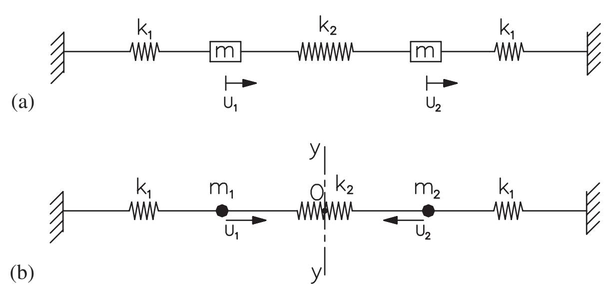

Figure 1. A 2 d.o.f.s spring–mass system: (a) original configuration and (b) transformed configuration with C_{1v} symmetry.

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connectivities of masses and springs are preserved, but very importantly, the transformed system now features an axis of symmetry y–y and a centre of symmetry O that were previously not there. In this new system, the positive direction of the d.o.f.s u_1 and u_2 of the two masses, previously towards the right (Figure 1a), now *becomes the direction towards the centre of symmetry O*. Thus the two freedoms u_1 and u_2 are now pointing inwards towards each other, giving the transformed configuration the symmetry of group C_{1v}.

The symmetry group C_{1v} has two symmetry elements: e (the identity element) and σ_y (reflection in the vertical plane y–y). The group is associated with two subspaces $S^{(1)}$ and $S^{(2)}$. In turn associated with each subspace $S^{(i)}$ $(i=1,2)$ is an idempotent $P^{(i)}$. For any symmetry group G, an idempotent is a linear combination of symmetry elements of the group, with the special property of being able to select, when applied upon the vector space V of a problem, all the vectors that belong to the subspace $S^{(i)}$ of the problem, and nullifying all the other vectors that do not belong to the subspace [10, 12, 13, 17]. In general, the idempotent $P^{(i)}$ (corresponding to a given subspace $S^{(i)}$ of a problem) can be written down directly from the character table of the symmetry group, using the relation [7, 10, 12]

$$P^{(i)} = \frac{\beta^{(i)}}{\eta} \sum_g \chi^{(i)}(g^{-1}) g \tag{5}$$

where η is the order of G (i.e. the number of elements of G); $\beta^{(i)}$ is the dimension of the ith irreducible representation, given by $\beta^{(i)} = \chi^{(i)}(e)$, the first character of the ith row of the character table; $\chi^{(i)}$ is a character of the ith irreducible representation; g is an element of G, and g^{-1} its inverse. For group C_{1v}, the idempotents for subspaces $S^{(1)}$ and $S^{(2)}$ are [1, 7, 12, 17]

$$P^{(1)} = \tfrac{1}{2}(e + \sigma_y) \tag{6a}$$

$$P^{(2)} = \tfrac{1}{2}(e - \sigma_y) \tag{6b}$$

respectively. Taking one idempotent at a time, and applying this to all the freedoms of the system, one obtains symmetry-adapted freedoms for the subspace in question. In general, these symmetry-adapted freedoms are not all independent. Any selection of a set of independent symmetry-adapted freedoms constitutes a set of basis vectors for that space.

By reference to the configuration of Figure 1(b), applying the idempotents $P^{(1)}$ and $P^{(2)}$ (Equation (6)) to the freedoms u_1 and u_2, we obtain

$$P^{(1)}u_1 = \tfrac{1}{2}(e + \sigma_y)u_1 = \tfrac{1}{2}(u_1 + u_2) = P^{(1)}u_2$$

$$P^{(2)}u_1 = \tfrac{1}{2}(e - \sigma_y)u_1 = \tfrac{1}{2}(u_1 - u_2) = -P^{(2)}u_2$$

and hence the basis vectors

$$\Phi^{(1)} = u_1 + u_2 \tag{7a}$$

$$\Phi^{(2)} = u_1 - u_2 \tag{7b}$$

for subspaces $S^{(1)}$ and $S^{(2)}$, respectively. Clearly, each of these subspaces is one dimensional (i.e. spanned by only one basis vector).

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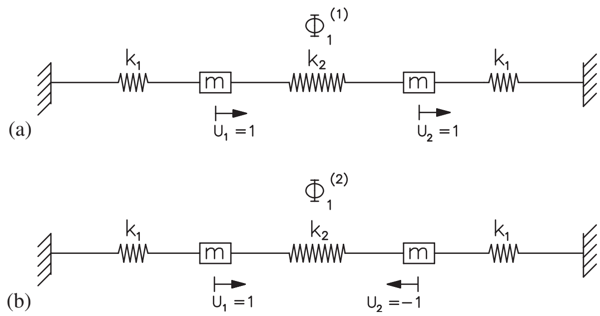

Figure 2. Application of displacement components of the basis vectors for the C_{1v} example: (a) subspace $S^{(1)}$ and (b) subspace $S^{(2)}$.

To obtain the required eigenvalues of the original problem, we consider each subspace of the problem, independently of the other subspace. Let us consider subspace $S^{(1)}$ first. To write down the stiffness coefficient(s) of the subspace, we now switch back to the original spring–mass system as depicted in Figure 1(a), where positive directions of the actual freedoms of the system are towards the right. Because $\Phi^{(1)}$ has both components u_1 and u_2 positive (Equation (7a)), we apply a unit positive displacement (i.e. pointing towards the right) *simultaneously* on each mass (m_1 and m_2), as depicted in Figure 2(a). The ensuing restoring force (i.e. in the direction opposite to that of the displacement being applied) on each mass is k_1 which, by definition, must be the stiffness coefficient. The mass value at the location of either component of $\Phi^{(1)}$ is m. Thus

$$\mathbf{K}^{(1)} = k_{11}^{(1)} = [k_1], \quad \mathbf{M}^{(1)} = m_{11}^{(1)} = [m] \tag{8}$$

Applying Equation (2), we obtain the first-degree characteristic equation

$$k_1 - \omega^2 m = 0 \tag{9}$$

giving the solution

$$\omega^2 = \frac{k_1}{m} \tag{10}$$

Similarly for subspace $S^{(2)}$, we apply the components of $\Phi^{(2)}$ (Equation (7b)) as a unit displacement in the positive direction for mass m_1 (i.e. $u_1 = 1$), and a simultaneous unit displacement in the *negative* direction for mass m_2 (i.e. $u_2 = -1$), as depicted in Figure 2(b). The ensuing restoring force (in the direction opposite to that of the imposed displacement) on each mass is $(k_1 + 2k_2)$. Thus

$$\mathbf{K}^{(2)} = k_{11}^{(2)} = [k_1 + 2k_2], \quad \mathbf{M}^{(2)} = m_{11}^{(2)} = [m] \tag{11}$$

Applying Equation (2), we obtain the first-degree characteristic equation

$$(k_1 + 2k_2) - \omega^2 m = 0 \tag{12}$$

with the solution

$$\omega^2 = \frac{k_1 + 2k_2}{m} \tag{13}$$

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The eigenvalues obtained independently for each subspace (Equations (10) and (13)) are, of course, the same eigenvalues as were obtained by conventional considerations (Equation (4)). Whereas the conventional approach required the solution of a quadratic characteristic equation, the group-theoretic method required the solution of two first-degree equations. For spring–mass systems which can be represented by equivalent graph configurations belonging to symmetry groups of higher order, the computational gains become more significant, and the group-theoretic approach becomes more rewarding. In the next section, we consider examples with more complex symmetry.

3. MORE COMPLEX MODELS WITH HIGHER-ORDER SYMMETRIES

3.1. A model with C_{3v} symmetry

Figure 3(a) depicts a 3 d.o.f. spring–mass system, with $m_1 = m_2 = m_3 = m$. The equivalent symmetric system is shown in Figure 3(b), where the configuration exhibits the symmetry properties of the symmetry group C_{3v}, whose elements are: e (identity); C_3 and C_3^{-1} (clockwise and anti-clockwise rotations, respectively, of $2\pi/3$ about the centre of symmetry O of the configuration); σ_1, σ_2, and σ_3 (reflections in the vertical planes $1-1$, $2-2$, and $3-3$). In the equivalent system, the positive direction of d.o.f.s u_1, u_2, and u_3 of the three masses, previously towards the right (Figure 3(a)), now becomes the direction towards the centre of symmetry O, preserving the C_{3v} symmetry.

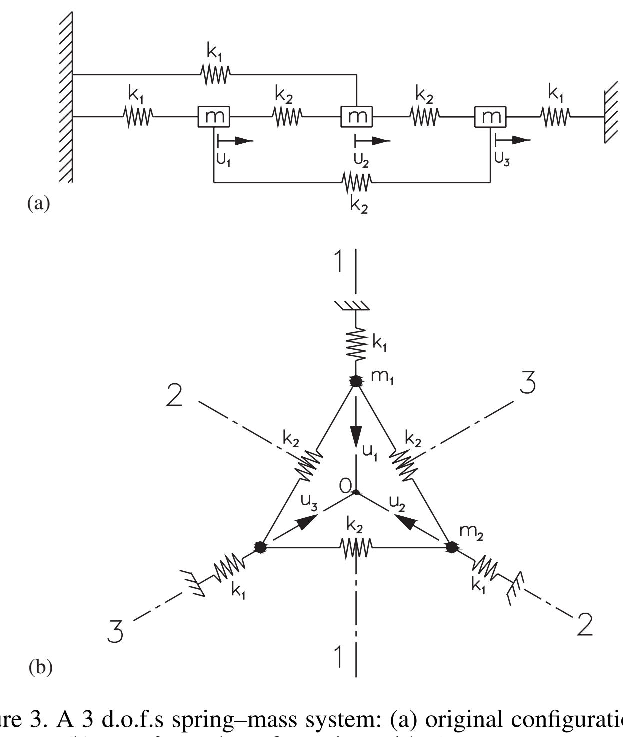

Figure 3. A 3 d.o.f.s spring–mass system: (a) original configuration and (b) transformed configuration with C_{3v} symmetry.

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For group C_{3v}, the three idempotents $P^{(1)}$, $P^{(2)}$, and $P^{(3)}$ corresponding to subspaces $S^{(1)}$, $S^{(2)}$, and $S^{(3)}$ are as follows [1, 7, 9, 13]:

$$P^{(1)} = \tfrac{1}{6}(e + C_3 + C_3^{-1} + \sigma_1 + \sigma_2 + \sigma_3) \tag{14a}$$

$$P^{(2)} = \tfrac{1}{6}(e + C_3 + C_3^{-1} - \sigma_1 - \sigma_2 - \sigma_3) \tag{14b}$$

$$P^{(3)} = \tfrac{1}{3}(2e - C_3 - C_3^{-1}) \tag{14c}$$

Applying these idempotents (one at a time) to the freedoms u_1, u_2, and u_3 in Figure 3(b), we obtain the basis vectors

Subspace $S^{(1)}$:

$$\Phi_1^{(1)} = u_1 + u_2 + u_3 \tag{15}$$

Subspace $S^{(3)}$:

$$\Phi_1^{(3)} = u_1 - \tfrac{1}{2}u_2 - \tfrac{1}{2}u_3 \tag{16a}$$

$$\Phi_2^{(3)} = -\tfrac{1}{2}u_1 + u_2 - \tfrac{1}{2}u_3 \tag{16b}$$

Thus subspace $S^{(1)}$ is one-dimensional and subspace $S^{(3)}$ is two-dimensional; subspace $S^{(2)}$ is a null space.

Reverting to the original spring–mass system of Figure 3(a), let us consider subspace $S^{(1)}$, spanned by the basis vector $\Phi_1^{(1)}$ (Equation (15)). Applying a unit positive displacement (i.e. pointing towards the right) simultaneously on each mass m_1, m_2, and m_3 as depicted in Figure 4(a), the ensuing restoring force on each mass is seen to be equal to k_1. The mass value at the location of any of the components of $\Phi_1^{(1)}$ is equal to m. Thus

$$\mathbf{K}^{(1)} = k_{11}^{(1)} = [k_1], \quad \mathbf{M}^{(1)} = m_{11}^{(1)} = [m] \tag{17}$$

This leads to the first-degree characteristic equation

$$k_1 - \omega^2 m = 0 \tag{18}$$

giving the solution

$$\omega^2 = \frac{k_1}{m} \tag{19}$$

For subspace $S^{(3)}$, the restoring forces on each of the masses m_1, m_2, and m_3, resulting from the application of the displacement components making up the basis vectors $\Phi_1^{(3)}$ and $\Phi_2^{(3)}$ (Equation (16)), as shown in Figure 4(b), are summarized in Table I.

For the application of either $\Phi_1^{(3)}$ or $\Phi_2^{(3)}$, it is noted that all restoring forces are proportional to the u value prescribed to the mass in question. Taking unit positive value of u (i.e. $u = +1$), the

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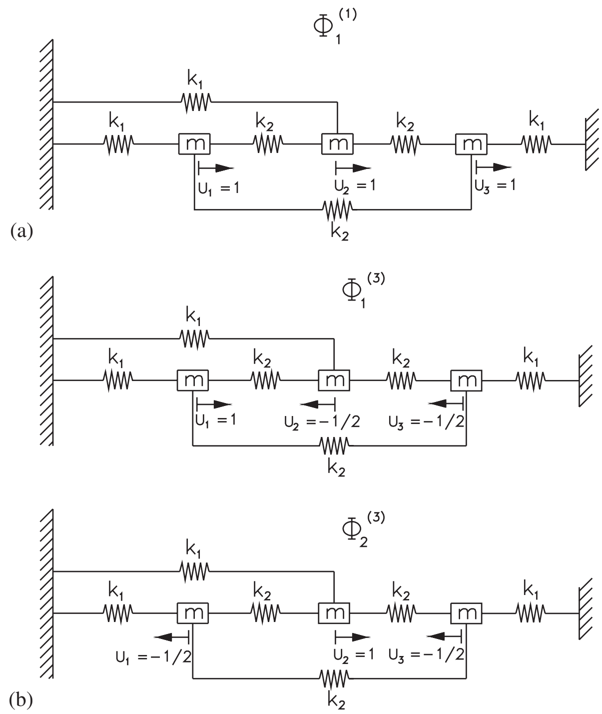

Figure 4. Application of displacement components of the basis vectors for the C_{3v} example: (a) subspace $S^{(1)}$ and (b) subspace $S^{(3)}$.

Table I. Restoring forces due to the application of the components of $\Phi_1^{(3)}$ and $\Phi_2^{(3)}$ for the C_{3v} example.

	$\Phi_1^{(3)}$		$\Phi_2^{(3)}$	
Mass	Displacement	Restoring force	Displacement	Restoring force
$m_1 = m$	$u_1 = +1.0$	$+1.0(k_1 + 3k_2)$	$u_1 = -0.5$	$-0.5(k_1 + 3k_2)$
$m_2 = m$	$u_2 = -0.5$	$-0.5(k_1 + 3k_2)$	$u_2 = +1.0$	$+1.0(k_1 + 3k_2)$
$m_3 = m$	$u_3 = -0.5$	$-0.5(k_1 + 3k_2)$	$u_3 = -0.5$	$-0.5(k_1 + 3k_2)$

corresponding restoring force is $(k_1 + 3k_2)$ in either case. Therefore,

$$\begin{aligned} k_{11}^{(3)} &= (k_1 + 3k_2), \quad k_{12}^{(3)} = 0 \\ k_{21}^{(3)} &= 0, \quad k_{22}^{(3)} = (k_1 + 3k_2) \end{aligned} \tag{20}$$

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Thus

$$\mathbf{K}^{(3)} = \begin{bmatrix} (k_1 + 3k_2) & 0 \\ 0 & (k_1 + 3k_2) \end{bmatrix}, \quad \mathbf{M}^{(3)} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \tag{21}$$

This leads to two uncoupled first-degree characteristic equations

$$(k_1 + 3k_2) - \omega^2 m = 0 \quad \text{(twice)} \tag{22}$$

giving the solutions

$$(\omega^2)_1^{(3)} = (\omega^2)_2^{(3)} = \frac{k_1 + 3k_2}{m} \tag{23}$$

The three natural circular frequencies of the original system are the collection of the eigenvalues yielded by the individual subspaces: one eigenvalue from subspace $S^{(1)}$, and two equal eigenvalues (a doubly repeating root) from subspace $S^{(3)}$. Thus, for the original system, and by reference to Equations (19) and (23)

$$\omega_1^2 = \frac{k_1}{m}, \quad \omega_2^2 = \omega_3^2 = \frac{k_1 + 3k_2}{m} \tag{24}$$

Instead of having to solve a third-degree characteristic equation as yielded by conventional considerations, the present group-theoretic approach involves the solution of only two separate first-degree characteristic equations, yielding all three required eigenvalues (one distinct and one repeated) of the original problem. This is clearly a significant reduction in computational effort. In general, repeating roots are associated with subspaces corresponding to those irreducible representations of a symmetry group whose dimension $\beta^{(i)}$ (as defined for Equation (5)) is greater than 1, which means that subspaces yielding repeating roots are known right from the outset.

3.2. A model with C_{6v} symmetry

Figure 5(a) shows a 6 d.o.f. spring–mass system, with $m_i = m$ for $i = 1, 2, \ldots, 6$. The equivalent symmetric system is shown in Figure 5(b). This configuration has a centre of symmetry O, and the symmetry elements: e (identity); C_6 and C_6^{-1} (clockwise and anticlockwise rotations, respectively, of $2\pi/6$ about the centre of symmetry O); C_3 and C_3^{-1} (clockwise and anticlockwise rotations, respectively, of $2\pi/3$ about the centre of symmetry O); C_2 (rotation of π about the centre of symmetry O); σ_a, σ_b, σ_c, σ_1, σ_2, and σ_3 (reflections in the vertical planes $A-A$, $B-B$, $C-C$, $1-1$, $2-2$, and $3-3$, respectively). The configuration of Figure 5(b) therefore belongs to the symmetry group C_{6v} of order 12. The positive direction of the freedoms u_1, u_2, u_3, u_4, u_5, and u_6 maps into the direction towards the centre of symmetry O.

The six idempotents of symmetry group C_{6v}, corresponding to the subspaces $S^{(1)}$, $S^{(2)}$, $S^{(3)}$, $S^{(4)}$, $S^{(5)}$, and $S^{(6)}$, are as follows [1, 7, 9, 13]:

$$P^{(1)} = \tfrac{1}{12}(e + C_6 + C_6^{-1} + C_3 + C_3^{-1} + C_2 + \sigma_a + \sigma_b + \sigma_c + \sigma_1 + \sigma_2 + \sigma_3) \tag{25a}$$

$$P^{(2)} = \tfrac{1}{12}(e + C_6 + C_6^{-1} + C_3 + C_3^{-1} + C_2 - \sigma_a - \sigma_b - \sigma_c - \sigma_1 - \sigma_2 - \sigma_3) \tag{25b}$$

$$P^{(3)} = \tfrac{1}{12}(e - C_6 - C_6^{-1} + C_3 + C_3^{-1} - C_2 + \sigma_a + \sigma_b + \sigma_c - \sigma_1 - \sigma_2 - \sigma_3) \tag{25c}$$

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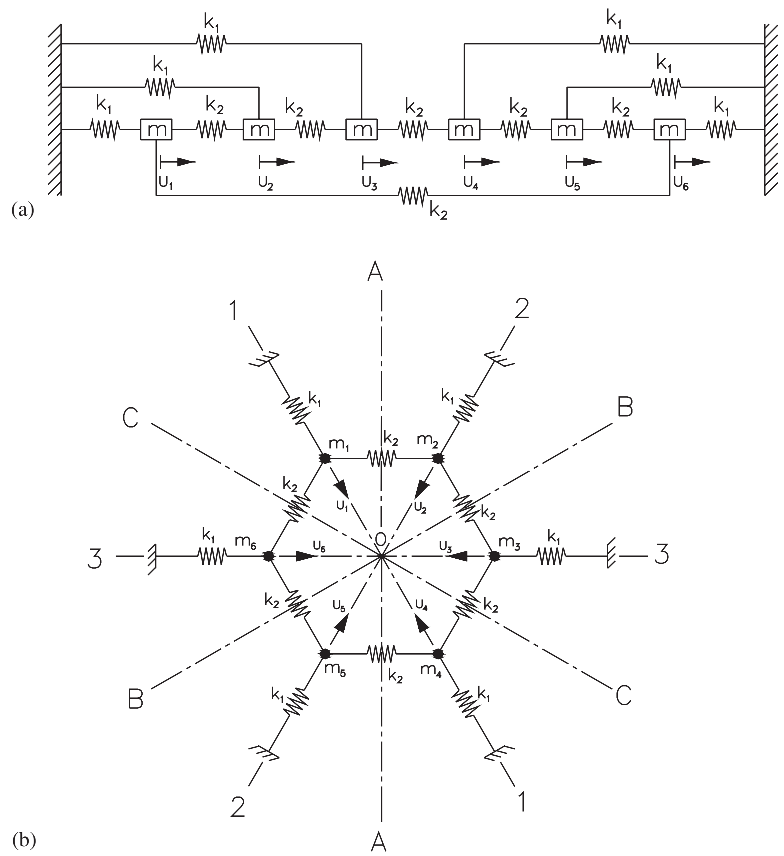

Figure 5. A 6 d.o.f.s spring–mass system: (a) original configuration and (b) transformed configuration with C_{6v} symmetry.

$$P^{(4)} = \tfrac{1}{12}(e - C_6 - C_6^{-1} + C_3 + C_3^{-1} - C_2 - \sigma_a - \sigma_b - \sigma_c + \sigma_1 + \sigma_2 + \sigma_3) \tag{25d}$$

$$P^{(5)} = \tfrac{1}{6}(2e + C_6 + C_6^{-1} - C_3 - C_3^{-1} - 2C_2) \tag{25e}$$

$$P^{(6)} = \tfrac{1}{6}(2e - C_6 - C_6^{-1} - C_3 - C_3^{-1} + 2C_2) \tag{25f}$$

Applying each idempotent to all the six freedoms u_i $(i=1,2,\ldots,6)$ of the configuration of Figure 5(b), and selecting the basis vectors for the subspace in question, we obtain the results

Subspace $S^{(1)}$:

$$\Phi_1^{(1)} = u_1 + u_2 + u_3 + u_4 + u_5 + u_6 \tag{26}$$

Subspace $S^{(4)}$:

$$\Phi_1^{(4)} = u_1 - u_2 + u_3 - u_4 + u_5 - u_6 \tag{27}$$

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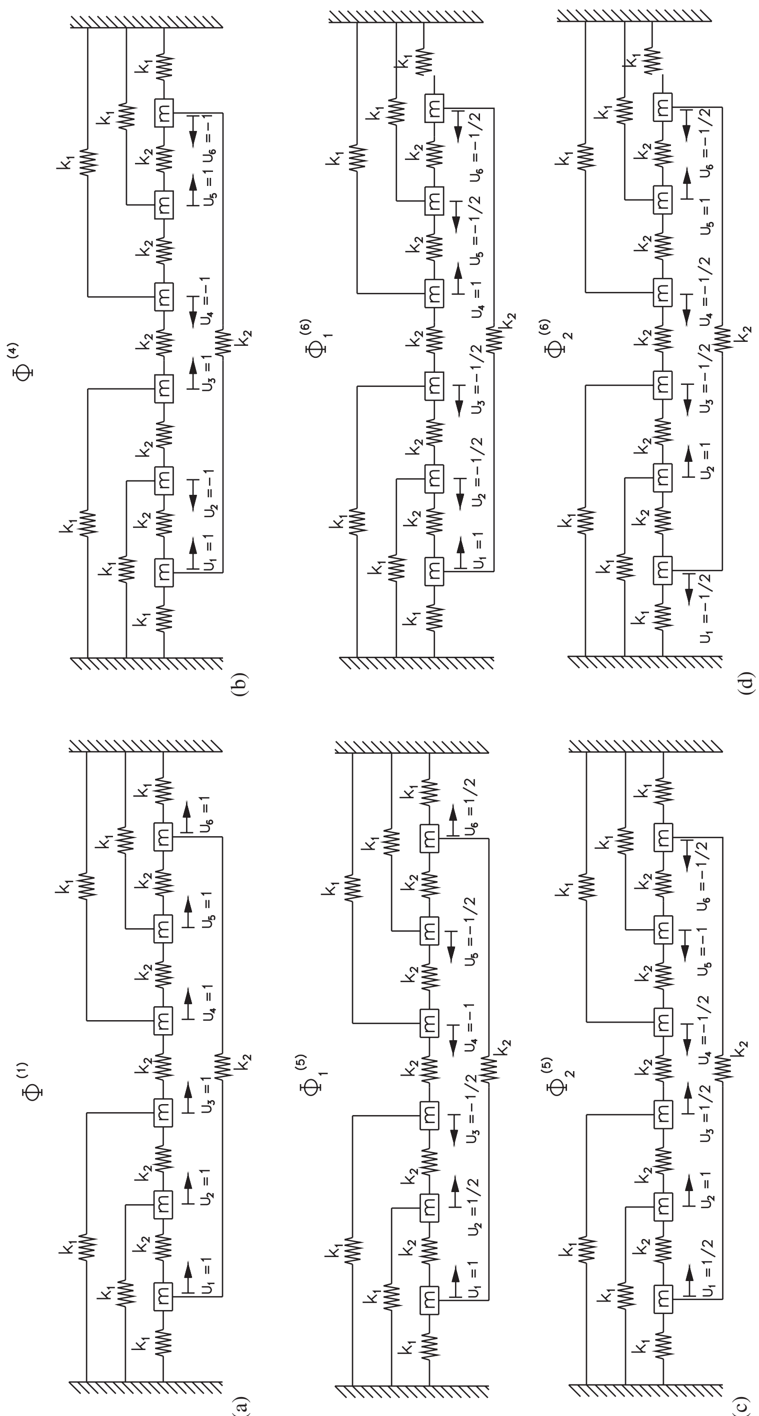

Figure 6. Application of displacement components of the basis vectors for the C_{6v} example: (a) subspace $S^{(1)}$; (b) subspace $S^{(4)}$; (c) subspace $S^{(5)}$; and (d) subspace $S^{(6)}$.

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Subspace $S^{(5)}$:

$$\Phi_1^{(5)} = u_1 + \tfrac{1}{2}u_2 - \tfrac{1}{2}u_3 - u_4 - \tfrac{1}{2}u_5 + \tfrac{1}{2}u_6 \tag{28a}$$

$$\Phi_2^{(5)} = \tfrac{1}{2}u_1 + u_2 + \tfrac{1}{2}u_3 - \tfrac{1}{2}u_4 - u_5 - \tfrac{1}{2}u_6 \tag{28b}$$

Subspace $S^{(6)}$:

$$\Phi_1^{(6)} = u_1 - \tfrac{1}{2}u_2 - \tfrac{1}{2}u_3 + u_4 - \tfrac{1}{2}u_5 - \tfrac{1}{2}u_6 \tag{29a}$$

$$\Phi_2^{(6)} = -\tfrac{1}{2}u_1 + u_2 - \tfrac{1}{2}u_3 - \tfrac{1}{2}u_4 + u_5 - \tfrac{1}{2}u_6 \tag{29b}$$

Thus subspaces $S^{(1)}$ and $S^{(4)}$ are one-dimensional, subspaces $S^{(5)}$ and $S^{(6)}$ are two-dimensional, while subspaces $S^{(2)}$ and $S^{(3)}$ are null spaces.

For subspace $S^{(1)}$, and considering $\Phi_1^{(1)}$ (Equation (26)), applying a unit positive displacement simultaneously on each mass m_i $(i = 1, 2, \ldots, 6)$ as depicted in Figure 6(a), the ensuing restoring force on each mass is seen to be equal to k_1. Thus

$$\mathbf{K}^{(1)} = k_{11}^{(1)} = [k_1], \quad \mathbf{M}^{(1)} = m_{11}^{(1)} = [m] \tag{30}$$

leading to the first-degree characteristic equation

$$k_1 - \omega^2 m = 0 \tag{31}$$

and the solution

$$\omega^2 = \frac{k_1}{m} \tag{32}$$

For subspace $S^{(4)}$, and considering $\Phi_1^{(4)}$ (Equation (27)), applying a unit displacement simultaneously on each mass, in the directions shown in Figure 6(b), results in a restoring force of $(k_1 + 4k_2)$ on each mass. Thus

$$\mathbf{K}^{(4)} = k_{11}^{(4)} = [k_1 + 4k_2], \quad \mathbf{M}^{(4)} = m_{11}^{(4)} = [m] \tag{33}$$

which leads to the first-degree characteristic equation

$$(k_1 + 4k_2) - \omega^2 m = 0 \tag{34}$$

and the solution

$$\omega^2 = \frac{k_1 + 4k_2}{m} \tag{35}$$

For subspace $S^{(5)}$, the restoring forces on each of the masses m_i $(i = 1, 2, \ldots, 6)$, resulting from the application of the displacement components making up the basis vectors $\Phi_1^{(5)}$ and $\Phi_2^{(5)}$ (Equations (28)) as shown in Figure 6(c), are summarized in Table II. Similarly for subspace $S^{(6)}$, the restoring forces on each of the masses m_i, resulting from the application of the displacement components making up the basis vectors $\Phi_1^{(6)}$ and $\Phi_2^{(6)}$ (Equations (29)) as shown in Figure 6(d), are summarized in Table III.

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Table II. Restoring forces due to the application of the components of $\Phi_1^{(5)}$ and $\Phi_2^{(5)}$ for the C_{6v} example.

	$\Phi_1^{(5)}$		$\Phi_2^{(5)}$	
Mass	Displacement	Restoring force	Displacement	Restoring force
$m_1 = m$	$u_1 = +1.0$	$+1.0(k_1+k_2)$	$u_1 = +0.5$	$+0.5(k_1+k_2)$
$m_2 = m$	$u_2 = +0.5$	$+0.5(k_1+k_2)$	$u_2 = +1.0$	$+1.0(k_1+k_2)$
$m_3 = m$	$u_3 = -0.5$	$-0.5(k_1+k_2)$	$u_3 = +0.5$	$+0.5(k_1+k_2)$
$m_4 = m$	$u_4 = -1.0$	$-1.0(k_1+k_2)$	$u_4 = -0.5$	$-0.5(k_1+k_2)$
$m_5 = m$	$u_5 = -0.5$	$-0.5(k_1+k_2)$	$u_5 = -1.0$	$-1.0(k_1+k_2)$
$m_6 = m$	$u_6 = +0.5$	$+0.5(k_1+k_2)$	$u_6 = -0.5$	$-0.5(k_1+k_2)$

Table III. Restoring forces due to the application of the components of $\Phi_1^{(6)}$ and $\Phi_2^{(6)}$ for the C_{6v} example.

	$\Phi_1^{(6)}$		$\Phi_2^{(6)}$	
Mass	Displacement	Restoring force	Displacement	Restoring force
$m_1 = m$	$u_1 = +1.0$	$+1.0(k_1+3k_2)$	$u_1 = -0.5$	$-0.5(k_1+3k_2)$
$m_2 = m$	$u_2 = -0.5$	$-0.5(k_1+3k_2)$	$u_2 = +1.0$	$+1.0(k_1+3k_2)$
$m_3 = m$	$u_3 = -0.5$	$-0.5(k_1+3k_2)$	$u_3 = -0.5$	$-0.5(k_1+3k_2)$
$m_4 = m$	$u_4 = +1.0$	$+1.0(k_1+3k_2)$	$u_4 = -0.5$	$-0.5(k_1+3k_2)$
$m_5 = m$	$u_5 = -0.5$	$-0.5(k_1+3k_2)$	$u_5 = +1.0$	$+1.0(k_1+3k_2)$
$m_6 = m$	$u_6 = -0.5$	$-0.5(k_1+3k_2)$	$u_6 = -0.5$	$-0.5(k_1+3k_2)$

Considering subspace $S^{(5)}$, and for the application of either $\Phi_1^{(5)}$ or $\Phi_2^{(5)}$, it is noted (from Table II) that all restoring forces are proportional to the u value prescribed to the mass in question. Taking unit positive value of u (i.e. $u = +1$), the corresponding restoring force is (k_1+k_2) in either case. Therefore,

$$\begin{aligned} k_{11}^{(5)} &= (k_1+k_2), \quad k_{12}^{(5)} = 0 \\ k_{21}^{(5)} &= 0, \quad k_{22}^{(5)} = (k_1+k_2) \end{aligned} \tag{36}$$

Thus

$$\mathbf{K}^{(5)} = \begin{bmatrix} (k_1+k_2) & 0 \\ 0 & (k_1+k_2) \end{bmatrix}, \quad \mathbf{M}^{(5)} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \tag{37}$$

This leads to two uncoupled first-degree characteristic equations

$$(k_1+k_2) - \omega^2 m = 0 \quad \text{(twice)} \tag{38}$$

giving the solutions

$$(\omega^2)_1^{(5)} = (\omega^2)_2^{(5)} = \frac{k_1+k_2}{m} \tag{39}$$

Similarly for subspace $S^{(6)}$, for the application of either $\Phi_1^{(6)}$ or $\Phi_2^{(6)}$, it is noted (from Table III) that all restoring forces are also proportional to the u value prescribed to the mass in question.

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Taking unit positive value of u (i.e. $u = +1$), the corresponding restoring force is $(k_1 + 3k_2)$ in either case. Hence,

$$\begin{aligned} k_{11}^{(6)} &= (k_1 + 3k_2), \quad k_{12}^{(6)} = 0 \\ k_{21}^{(6)} &= 0, \quad k_{22}^{(6)} = (k_1 + 3k_2) \end{aligned} \tag{40}$$

Thus

$$\mathbf{K}^{(6)} = \begin{bmatrix} (k_1 + 3k_2) & 0 \\ 0 & (k_1 + 3k_2) \end{bmatrix}, \quad \mathbf{M}^{(6)} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \tag{41}$$

leading to two uncoupled first-degree characteristic equations

$$(k_1 + 3k_2) - \omega^2 m = 0 \quad \text{(twice)} \tag{42}$$

with the solutions

$$(\omega^2)_1^{(6)} = (\omega^2)_2^{(6)} = \frac{k_1 + 3k_2}{m} \tag{43}$$

Thus, in summary, subspaces $S^{(1)}$ and $S^{(4)}$ yielded one eigenvalue each; subspaces $S^{(5)}$ and $S^{(6)}$ each yielded two equal eigenvalues (doubly repeating roots). Collecting the six eigenvalues thus obtained, the six natural circular frequencies of the original system (in ascending order) are as follows:

$$\omega_1^2 = \frac{k_1}{m}, \quad \omega_2^2 = \omega_3^2 = \frac{k_1 + k_2}{m}, \quad \omega_4^2 = \omega_5^2 = \frac{k_1 + 3k_2}{m}, \quad \omega_6^2 = \frac{k_1 + 4k_2}{m} \tag{44}$$

Instead of having to expand a 6×6 determinant and solve a sixth-degree characteristic polynomial as yielded by conventional considerations, the present group-theoretic approach required the solution of only four separate *first-degree* characteristic equations, yielding all six eigenvalues (two distinct roots, and two pairs of repeated roots) of the original problem. Here, we see that the computational effort of the group-theoretic procedure is less than 10% of that associated with the conventional analysis.

4. CONCLUDING REMARKS

In this article, it is shown how unsymmetrical (or weakly symmetrical) spring–mass dynamic systems can be transformed into equivalent dynamic systems featuring the maximum possible symmetry, and then a group-theoretic procedure employed to calculate all the eigenvalues (i.e. natural circular frequencies) of the systems. Unlike some existing group-theoretic algorithms, the present method does not require the assembly of the full stiffness matrix of the entire system first; it is only necessary to compute much smaller matrices (within the independent subspaces of the decomposed problem), on the basis of which all the eigenvalues of the original system are generated. The proposed approach will therefore generally result in substantial reductions in computational effort, particularly in the case of problems involving a large number of d.o.f.s, and associated with symmetry groups of high order. ‘Triangular’ symmetries of the type associated with groups C_{3v} and C_{6v} are particularly difficult to exploit using conventional methods. It is shown how group theory reveals the existence of repeating roots for such configurations, and the symmetry types (i.e. subspaces) to which such repeated frequencies belong, as well as simplifying the actual computation of these frequencies.

REFERENCES

1. Hamermesh M. *Group Theory and its Application to Physical Problems*. Pergamon Press: Oxford, 1962.
2. Wigner EP. *Group Theory and its Applications to the Quantum Mechanics of Atomic Spectra*. Academic Press: New York, 1959.
3. Tinkham M. *Group Theory and Quantum Mechanics*. McGraw-Hill: New York, 1975.
4. Schonland D. *Molecular Symmetry*. Van Nostrand: London, 1965.
5. Wherrett BS. *Group Theory for Atoms*, *Molecules and Solids*. Prentice-Hall: Englewood Cliffs, NJ, 1986.
6. Healey TJ. A group-theoretic approach to computational bifurcation problems with symmetry. *Computer Methods in Applied Mechanics and Engineering* 1988; **67**:257–295.
7. Zlokovic GM. *Group Theory and G-Vector Spaces in Structural Analysis*. Ellis Horwood: Chichester, U.K., 1989.
8. Ikeda K, Murota K. Bifurcation analysis of symmetric structures using block-diagonalisation. *Computer Methods in Applied Mechanics and Engineering* 1991; **86**:215–243.
9. Zingoni A, Pavlovic MN, Zlokovic GM. A symmetry-adapted flexibility approach for multi-storey space frames: general outline and symmetry-adapted redundants. *Structural Engineering Review* 1995; **7**(2):107–119.
10. Zingoni A. Group-theoretic applications in solid and structural mechanics: a review. In *Computational Structures Technology*, Topping BHV, Bittnar Z (eds). Saxe-Coburg Publications: Stirling, Scotland, 2002; 283–317.
11. Healey TJ, Treacy JA. Exact block diagonalisation of large eigenvalue problems for structures with symmetry. *International Journal for Numerical Methods in Engineering* 1991; **31**:265–285.
12. Zingoni A. An efficient computational scheme for the vibration analysis of high-tension cable nets. *Journal of Sound and Vibration* 1996; **189**(1):55–79.
13. Zingoni A. On the symmetries and vibration modes of layered space grids. *Engineering Structures* 2005; **27**(4):629–638.
14. Kaveh A, Sayarinejad MA. Eigensolutions for matrices of special patterns. *Communications in Numerical Methods in Engineering* 2003; **19**:125–136.
15. Kaveh A, Sayarinejad MA. Graph symmetry in dynamic systems. *Computers and Structures* 2004; **82**:2229–2240.
16. Kaveh A, Nikbakht M. Decomposition of symmetric mass–spring vibrating systems using groups, graphs and linear algebra. *Communications in Numerical Methods in Engineering* 2007, in press.
17. Zingoni A. A group-theoretic formulation for symmetric finite elements. *Finite Elements in Analysis and Design* 2005; **41**(6):615–635.

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