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On group-theoretic eigenvalue vibration analysis of structural systems with C_{6v} symmetry

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ABSTRACT

In considerations of the linear vibration of symmetric systems, group theory allows the space of the eigenvalue problem to be decomposed into independent subspaces that are spanned by symmetry-adapted freedoms. These problems usually feature one or more *degenerate* subspaces (i.e. subspaces that contain repeating solutions). For such subspaces, the associated idempotents, as calculated from the character table of the symmetry group, are not capable of full decomposition of the subspace. In this paper, and based on group theory, simple algebraic operators that fully decompose the two degenerate subspaces of structural problems belonging to the C_{6v} symmetry group are proposed. The operators are applied to the vibration of a spring-mass system, for which the results for natural frequencies are found to agree exactly with results from the literature. Their application to the vibration of a hexagonal plane grid reveals new insights on the character of the modes of degenerate subspaces. The overall conclusion is that, for problems belonging to the C_{6v} symmetry group, the proposed operators allow the mixed modes of degenerate subspaces to be separated into two distinct symmetry categories, and are very effective in simplifying the actual computation of the repeating eigenvalues of these subspaces.

1. Introduction

It is well known that the free vibration of physical systems with symmetry is characterised by phenomena such as repetition of natural frequencies, the occurrence of stationary points along specific lines or in specific planes, and in general, patterns of vibration (mode shapes) that reflect all or part of the symmetry of the physical system [1–3]. While symmetry may be taken advantage of in simplifying the analysis of the system, it also attracts complications in structural behaviour, such as the occurrence of multiple critical points in bifurcation analysis [4,5], and the clustering of eigenvalues in problems of the vibration or buckling of structures. Such phenomena introduce problems of numerical ill-conditioning of solution procedures. The mathematics of group theory is well-suited to the study of the properties and behaviour of physical systems with symmetry [6]. It is particularly advantageous for systems that have complex symmetry which cannot be taken into account using conventional considerations of symmetry (such as behaviour across planes of symmetry and planes of anti-symmetry).

Within physics and chemistry, applications of group theory date back many years, and have included the study of problems in crystallography, quantum mechanics and molecular symmetry [7–9], among others. Engineering applications only came later, with problems in structural mechanics only receiving significant attention over the past 40 years. Notable among these have been studies of the bifurcation of space trusses and frameworks [5,10,11], the structural analysis of rigidly-jointed space frames [12] and pin-jointed

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trusses [13], the vibration analysis of cable nets [1,14], space grids [2] and plates [15], the rigidity and kinematics of skeletal structures [16,17], and the buckling of frames and other skeletal structures [18–21]. More recently, the stability and vibration of origami structures has also received attention [22,23]. Another interesting area of application of group theory has been the deformation of auxetic metamaterials with a triangular internal structure [24] and the buckling of "architectured" materials with a hexagonal/honeycomb internal structure [25].

Group theory affords both quantitative and qualitative benefits, namely (i) the reduction of computational effort, and (ii) the provision of deeper insights on structural behaviour [3,26], and a better understanding of physical phenomena such as the occurrence of repeating natural frequencies [1,27,28]. For a symmetric structural system undergoing vibrations, group theory also allows the symmetries of all modes of vibration, and the location of all stationary points (i.e. points that do not move during vibration), to be predicted before any detailed numerical computations are performed [2,3,26,29,30]. The key to group-theoretic simplification is the decomposition of the space of the symmetric problem into independent subspaces that are spanned by symmetry-adapted variables, allowing the problem to be split into smaller independent problems that are easier to study and analyse. By separating the computation of closely-spaced or coincident eigenvalues into independent subspaces, group theory also circumvents the numerical problems associated with computing such solutions in the full space of the problem.

Group-theoretic decomposition may be effected by transforming the normal functions in the vector space of the problem into symmetry-adapted functions, which become the variables in the subspaces. This transformation may be achieved by means of idempotents (or projection operators) of the symmetry group [6], which are linear combinations (in a very specific way) of the symmetry elements of the group. However, for certain symmetry groups, some of the idempotents do not sufficiently decompose the associated subspace. These subspaces typically contain repeating solutions (which, in the case of eigenvalue vibration problems, are repeating natural frequencies); in the present considerations, such subspaces will be referred to as *degenerate* subspaces. The degeneracy referred to in the present contribution is that related to the symmetry properties of a physical configuration. It should be noted that there is also another type of degeneracy not related to symmetry, which is often referred to as "accidental degeneracy"; this will not be considered.

For degenerate subspaces (these will be explained more fully in the next section), repeating solutions remain within the same subspace after the normal process of group-theoretic decomposition afforded by standard idempotents. This makes their numerical computation problematic. Furthermore, and with reference to eigenvalue vibration problems, the symmetries of the modes populating degenerate subspaces are not of the same type, and yet we expect modes belonging to the same subspace to have the same symmetry type. This clearly points to the need for further decomposition of degenerate subspaces, if the means for achieving this can be found.

Symmetry groups that are associated with degenerate subspaces include cyclic (C) and dihedral (D) groups, where some solutions repeat twice, and higher-order groups of the tetrahedral (T), octahedral (O) and icosahedral (I) type, where solutions can repeat three or more times. With structural, mechanical and aerospace engineering applications in mind, we will limit our considerations to point groups, which describe the symmetry of configurations with one centre of symmetry, i.e. the point where all axes of rotation and all reflection planes intersect; we will not be concerned with translational symmetries. In the present work, we will focus on the symmetry group C_{6v} describing the symmetry of a regular hexagon. Structural configurations that belong to this symmetry group include cable nets, lattice shells, truss domes, plane grids and space grids for roofing applications, as well as radio telescopes, antennae and deployable satellite structures.

General formulations for deriving the projection operators of degenerate subspaces may be seen in early texts on applications of group theory in physics and chemistry (see, for example, the text of McWeeny [31]), and may also be seen in some of the later work related to structural eigenvalue problems [32], but those approaches require matrices of the irreducible representations of the symmetry group to be written down first. In this contribution, we will propose simple algebraic operators that further decompose the two degenerate subspaces of symmetry group C_{6v} without the need to derive matrices of the irreducible representations of the symmetry group. These operators will be in terms of the symmetry elements of the group, and presented in a form that is easy to implement from a practical engineering point of view.

While less general in the sense of being only applicable to problems that have C_{6v} symmetry (of which there are many such configurations in structural engineering), the proposed operators have the advantage of offering an alternative approach that is simpler to implement in practical engineering computations. They provide an effective tool for the separation of the vibration modes that occur in degenerate subspaces, at the same time making it easier to visualise the distinctive symmetries of the ensuing semi-subspaces (after decomposition). Application of these operators will be illustrated through two vibration problems, revealing the nature of the symmetries associated with degenerate subspaces, as well as other insights on structural vibration behaviour.

The structure of the rest of the paper is as follows. An overview of symmetry groups and their representation is given in Section 2, leading to the concepts of idempotents and subspaces. The treatment ends with a motivation for the present study. In Section 3, new operators for the full decomposition of the two degenerate subspaces of symmetry group C_{6v} are proposed. It is shown that these operators have all the essential properties of subspace idempotents. In Section 4, the proposed operators are applied to the vibration of a spring-mass system, for which the repeating natural frequencies are computed, and the results compared with existing results in the literature, thus validating the operators. In Section 5, to further illustrate the qualitative benefits of the new operators, the transverse vibrations of a 3-way hexagonal grid are considered, and all symmetries of the vibration modes elucidated. Concluding remarks are made in Section 6.

2. Symmetry groups, vector-space decomposition and motivation for the present study

A detailed theoretical treatment of symmetry groups and their representation may be seen in classical texts on the subject [6–9], while applications specifically related to the vibration of structural and mechanical systems may be seen in more recent texts [33,34]. Here, we will simply outline the essential results of the theory, using the symmetry group C_{6v} to illustrate the concepts, and making reference to eigenvalue vibration analysis where appropriate.

We consider a set of elements $\{a, b, c, \ldots, g, \ldots\}$ to constitute a group G with respect to binary operations if the following axioms are satisfied [6,33,34]:

(i) The product c of any two elements a and b of the group, denoted by $c = ab$, must be a unique element which also belongs to the group.
(ii) Among the elements of G, there must exist an identity element e which, when multiplied with any element a of the group, leaves the element unchanged: $ea = ae = a$.
(iii) For every element a of G, there must exist another element d also belonging to the group G, such that $ad = da = e$; d is referred to as the inverse of a, and denoted by a^{-1}.
(iv) How elements are grouped together in executing the multiplication of three or more elements of G does not affect the result (that is, multiplication is associative): $(ab)c = a(bc)$.

However, multiplication is not necessarily commutative (that is, the result $ab = ba$ does not need to hold). The group G is referred to as a *symmetry group* if all its elements are symmetry operations. For objects or systems which are finite, symmetry operations are typically of the following types:

(i) reflections in planes of symmetry; these are denoted by σ_l, where l is the plane of symmetry
(ii) rotations about an axis of symmetry; these are denoted by C_n if the angle of rotation is $2\pi/n$
(iii) rotation-reflections; these are denoted by S_n, and are a combination of a rotation through an angle $2\pi/n$ and a reflection in the plane perpendicular to the axis of rotation.

To illustrate symmetry elements, let us consider Fig. 1, which shows a double-layer regular hexagonal grid (in plan and elevation) supported at 6 joints in the bottom layer (the corners of the hexagon), and with 37 unsupported joints (13 in the bottom layer and 24 in the top layer). Joints of the bottom layer lie at the vertices of equilateral triangles, while joints of the upper layer coincide with the centroids of the bottom equilateral triangles. The overall configuration belongs to symmetry group C_{6v}, which characterises systems with the symmetries of a regular hexagon. The centre of symmetry in plan coincides with joint 7. With reference to the vertical axis of rotational symmetry passing through joint 7, it is evident that the configuration has (i) 6 rotational symmetries $\{e, C_6, C_6^{-1}, C_3, C_3^{-1}, C_2\}$ (which are: the identity element (equivalent to a zero rotation, or a complete rotation through 2π); rotations of 2π /6 clockwise and $2\pi/6$ anticlockwise; rotations of $2\pi/3$ clockwise and $2\pi/3$ anticlockwise; a rotation of $2\pi/2$ clockwise or anticlockwise), and (ii) 6 reflection symmetries $\{\sigma_a, \sigma_b, \sigma_c, \sigma_1, \sigma_2, \sigma_3\}$ (which are: reflections in the vertical planes $A-A$, $B-B$, $C-C$ through the midsides of

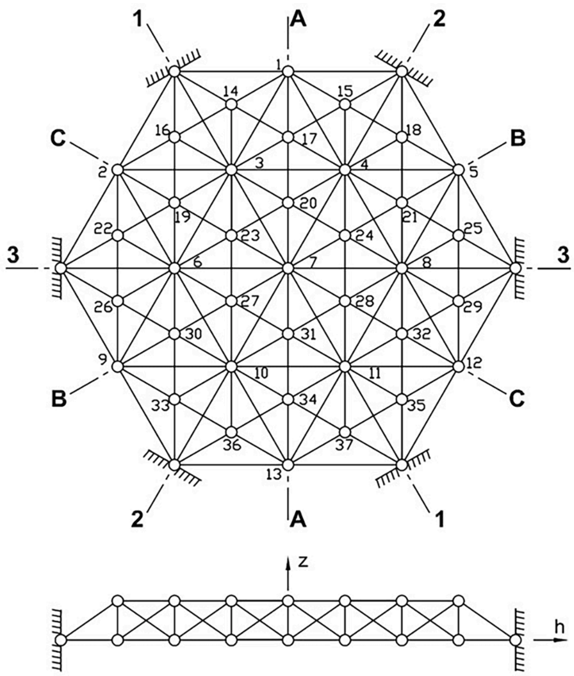


Fig. 1. Hexagonal grid with C_{6v} symmetry [2].

the hexagon; reflections in the vertical planes $1-1$, $2-2$, $3-3$ through the corners of the hexagon). Symmetry group C_{6v} is said to be of order 12 because it has 12 symmetry elements in total.

Fig. 2 shows, for a regular hexagon with symmetry axes $A-A$, $B-B$, $C-C$, $1-1$, $2-2$ and $3-3$, a point P arbitrarily positioned on the top side of the hexagon. The results of moving point P through the six rotation operations $\{e,\ C_6,\ C_6^{-1},\ C_3,\ C_3^{-1},\ C_2\}$ are as shown in red, while those of moving point P through the six reflection operations $\{\sigma_a, \sigma_b,\ \sigma_c,\ \sigma_1,\ \sigma_2,\ \sigma_3\}$ are shown in black. Note that the identity operation e is considered as a rotation though an angle of zero or 2π.

The permutations in Fig. 2 allow us to construct the multiplication table (or group table) of symmetry group C_{6v}, as shown in Table 1. The 144 elements of the table are the results of multiplying a symmetry element α on the left side of the table by a symmetry element β at the top of the table, the multiplication being in the order $\alpha\beta$. Note that the group table of symmetry group C_{6v} is not symmetric, since the order of the multiplication affects the result (i.e. $\alpha\beta \neq \beta\alpha$ in general). The following properties are also worth noting: (i) every symmetry element of the group appears only once in a given row and only once in a given column; (ii) the product of two rotations is a rotation (36 results); (iii) the product of two reflections is a rotation (36 results); (iv) the product of a reflection and a rotation, or a rotation and a reflection, is a reflection (72 results).

The set of all results obtained by evaluating $\alpha^{-1}\beta\alpha$ for all symmetry elements α of the group is referred to as the class of β. Every element of the symmetry group appears in only one class L (there is no overlap), and the class of the identity element contains only the identity element. Derivations of the classes of some simple C_{nv} groups may be seen in Ref. [34]. Symmetry group C_{6v} may be shown to have six classes, which are as follows:

$$L_1 = \{E\};\ L_2 = \{C_6,\ C_6^{-1}\};\ L_3 = \{C_3,\ C_3^{-1}\};\ L_4 = \{C_2\};\ L_5 = \{\sigma_a,\ \sigma_b,\ \sigma_c\};\ L_6 = \{\sigma_1,\ \sigma_2,\ \sigma_3\}$$

Symmetry operations may be represented by $n \times n$ transformation matrices in an n-dimensional vector space. Reducible representations are those whose matrices can be written as a sum of matrices of lower dimension by changing the basis of the vector space, while irreducible representations (which we will denote by Γ) are those whose matrices cannot be written as a sum of matrices of lower dimension. The number of irreducible representations of a symmetry group is equal to the number of classes that the symmetry group has. Each irreducible representation is associated with a particular type of symmetry that characterises an independent symmetry subspace S of the problem. If the symmetry group has k classes, then the number of irreducible representations will be k, and the

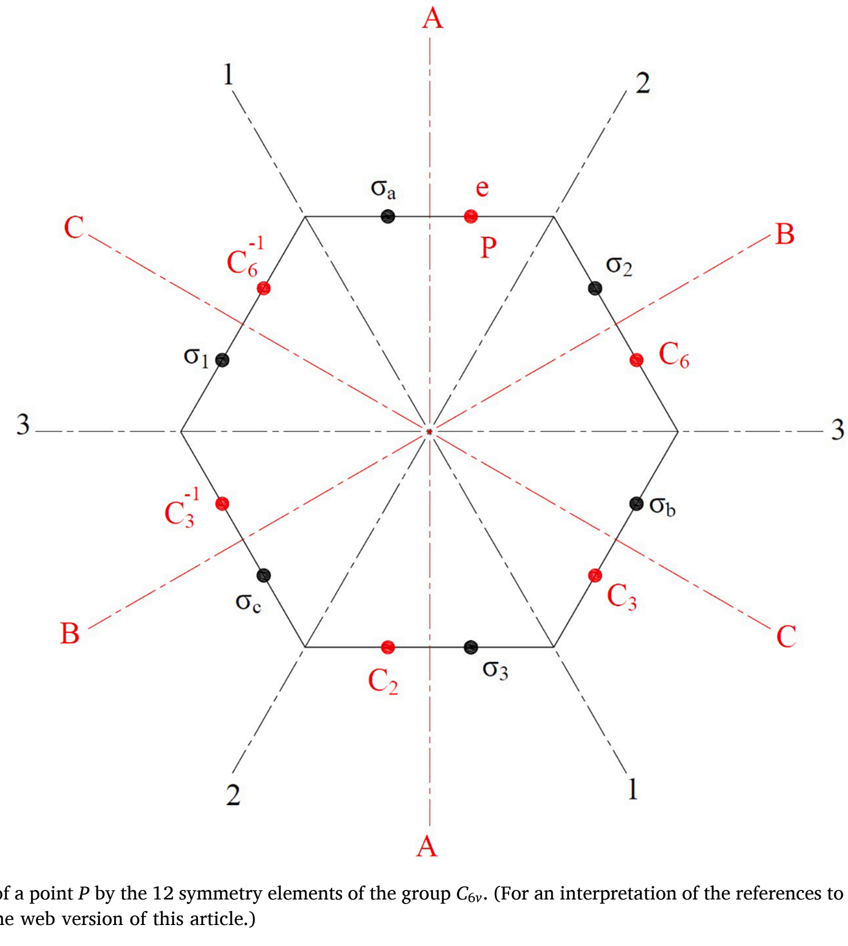


Fig. 2. Permutations of a point P by the 12 symmetry elements of the group C_{6v}. (For an interpretation of the references to colour in this figure, the reader is referred to the web version of this article.)

Table 1
Multiplication table of symmetry group C_{6v}.

	e	C_6	C_6^{-1}	C_3	C_3^{-1}	C_2	σ_a	σ_b	σ_c	σ_1	σ_2	σ_3
e	e	C_6	C_6^{-1}	C_3	C_3^{-1}	C_2	σ_a	σ_b	σ_c	σ_1	σ_2	σ_3
C_6	C_6	C_3	e	C_2	C_6^{-1}	C_3^{-1}	σ_1	σ_2	σ_3	σ_c	σ_a	σ_b
C_6^{-1}	C_6^{-1}	e	C_3^{-1}	C_6	C_2	C_3	σ_2	σ_3	σ_1	σ_a	σ_b	σ_c
C_3	C_3	C_2	C_6	C_3^{-1}	e	C_6^{-1}	σ_c	σ_a	σ_b	σ_3	σ_1	σ_2
C_3^{-1}	C_3^{-1}	C_6^{-1}	C_2	e	C_3	C_6	σ_b	σ_c	σ_a	σ_2	σ_3	σ_1
C_2	C_2	C_3^{-1}	C_3	C_6^{-1}	C_6	e	σ_3	σ_1	σ_2	σ_b	σ_c	σ_a
σ_a	σ_a	σ_2	σ_1	σ_b	σ_c	σ_3	e	C_3	C_3^{-1}	C_6^{-1}	C_6	C_2
σ_b	σ_b	σ_3	σ_2	σ_c	σ_a	σ_1	C_3^{-1}	e	C_3	C_2	C_6^{-1}	C_6
σ_c	σ_c	σ_1	σ_3	σ_a	σ_b	σ_2	C_3	C_3^{-1}	e	C_6	C_2	C_6^{-1}
σ_1	σ_1	σ_a	σ_c	σ_2	σ_3	σ_b	C_6	C_2	C_6^{-1}	e	C_3	C_3^{-1}
σ_2	σ_2	σ_b	σ_a	σ_3	σ_1	σ_c	C_6^{-1}	C_6	C_2	C_3^{-1}	e	C_3
σ_3	σ_3	σ_c	σ_b	σ_1	σ_2	σ_a	C_2	C_6^{-1}	C_6	C_3	C_3^{-1}	e

number of independent symmetry subspaces of the problem is also expected to be k.

The trace of a matrix representing a symmetry operation β is called the character of β, denoted by $\zeta(\beta)$. For any representation of the symmetry group, symmetry elements which belong to the same class have the same character, since traces of conjugate elements are equal. Each irreducible representation of a symmetry group has a unique set of characters that remains the same even if the basis of the vector space of the problem is changed. Since elements belonging to the same class have the same character, we may show the characters for each irreducible representation $\Gamma^{(i)}$ ($i = 1, 2, \ldots, k$) as a compact table with k rows of irreducible representations and k columns of classes. The format of character tables is given in Table 2, where $L_1, L_2, \ldots, L_k$ denote the k different classes of the symmetry group G, while $\Gamma^{(1)}, \Gamma^{(2)}, \ldots, \Gamma^{(k)}$ denote the k irreducible representations of G. The orthogonality property of irreducible representations implies that any two rows of the character table, or any two columns of the character table, are orthogonal.

The character table of symmetry group C_{6v} is given by Table 3. Here, we use Mullikan's notation for irreducible representations, which denotes 1-dimensional representations by the symbols A and B, and 2-dimensional representations by the symbol E. In character tables, the dimension of the irreducible representation $\Gamma^{(i)}$ is simply given by the first character in the row for $\Gamma^{(i)}$, which is the character of the identity element. It is important to note the first four irreducible representations $\{A_1, A_2, B_1, B_2\}$ are 1-dimensional, while the last two representations $\{E_1, E_2\}$ are 2-dimensional. The implication of this is that the first four subspaces of eigenvalue problems belonging to symmetry group C_{6v}, denoted by $S^{(1)}, S^{(2)}, S^{(3)}$ and $S^{(4)}$, will have distinct eigenvalues, while the last two subspaces, denoted by $S^{(5)}$ and $S^{(6)}$, will have doubly-repeating eigenvalues, and will be referred to as *degenerate* subspaces.

Each irreducible representation $\Gamma^{(i)}$ ($i = 1, 2, \ldots, k$) of the symmetry group is associated with a unique idempotent $P^{(i)}$ ($i = 1, 2, \ldots, k$), which is a very specific linear combination of the symmetry elements of the group. Idempotents satisfy the relation $P^{(i)}P^{(i)} = P^{(i)}$ for all i. More importantly, different idempotents are *orthogonal* to each other, i.e. $P^{(i)}P^{(j)} = 0$ if $i \neq j$. An idempotent $P^{(i)}$ has the special property of nullifying all functions that do not belong to the subspace $S^{(i)}$ of the problem, and selecting only functions that belong to subspace $S^{(i)}$. The selected functions will all have a definite symmetry type characteristic of subspace $S^{(i)}$, and are therefore referred to as *symmetry-adapted functions*. Thus, idempotents serve as projector operators [6] of the symmetry group.

The idempotents of a symmetry group G can be written down directly from the character table of the group, using the relation [6, 33,34]

$$P^{(i)} = \frac{h_i}{h} \sum_{\alpha} \zeta_i\left(\alpha^{-1}\right)\alpha \tag{1}$$

where $P^{(i)}$ corresponds to $\Gamma^{(i)}$ (the ith irreducible representation of the symmetry group G), h_i is the dimension of $\Gamma^{(i)}$ (given by $h_i = \zeta_i(e)$, the first character of the ith row of the character table), h is the order of the symmetry group (that is, the number of elements of G), ζ_i is a character of $\Gamma^{(i)}$, α is an element of G, and α^{-1} its inverse. If $\Gamma^{(i)}$ is a 1-dimensional representation of G, then $h_i = 1$; if $\Gamma^{(i)}$ is a 2-dimensional representation of G, then $h_i = 2$, and so forth.

It should be pointed out that idempotents may also be derived using irreducible representations rather than character tables. This approach has been followed by investigators like Healey [5], Ikeda & Murota [10], Combescure [11], and Kangwai *et al.* [13], among

Table 2
Format of character tables.

G	L_1	L_2	.	.	L_k
$\Gamma^{(1)}$	$\zeta_1^{(1)}$	$\zeta_2^{(1)}$	.	.	$\zeta_k^{(1)}$
$\Gamma^{(2)}$	$\zeta_1^{(2)}$	$\zeta_2^{(2)}$	.	.	$\zeta_k^{(2)}$
.	.	.	.	.	.
.	.	.	.	.	.
$\Gamma^{(k)}$	$\zeta_1^{(k)}$	$\zeta_2^{(k)}$	.	.	$\zeta_k^{(k)}$

Table 3
Character table of symmetry group C_{6v}.

C_{6v}	L_1 : $\{E\}$	L_2 : $\{C_6, C_6^{-1}\}$	L_3 : $\{C_3, C_3^{-1}\}$	L_4 : $\{C_2\}$	L_5 : $\{\sigma_a, \sigma_b, \sigma_c\}$	L_6 : $\{\sigma_1, \sigma_2, \sigma_3\}$
A_1	1	1	1	1	1	1
A_2	1	1	1	1	–1	–1
B_1	1	–1	1	–1	1	–1
B_2	1	–1	1	–1	–1	1
E_1	2	1	–1	–2	0	0
E_2	2	–1	–1	2	0	0

others. However, and consistent with the approach adopted by the author in all his previous work on group-theoretic applications in structural mechanics [1–3,12,14,15,34], the use of character tables is preferred, because character tables are readily available in the literature, and no matrices of irreducible representations need to be derived. This makes the derivation of idempotents a very simple process that engineers can easily implement without any detailed knowledge of representation theory.

Using Eq. (1) in conjunction with Table 3, the six idempotents of symmetry group C_{6v} are obtained as follows [2,3,27]:

$$P^{(1)} = \frac{1}{12}\left(e + C_6 + C_6^{-1} + C_3 + C_3^{-1} + C_2 + \sigma_a + \sigma_b + \sigma_c + \sigma_1 + \sigma_2 + \sigma_3\right) \tag{2}$$

$$P^{(2)} = \frac{1}{12}\left(e + C_6 + C_6^{-1} + C_3 + C_3^{-1} + C_2 - \sigma_a - \sigma_b - \sigma_c - \sigma_1 - \sigma_2 - \sigma_3\right) \tag{3}$$

$$P^{(3)} = \frac{1}{12}\left(e - C_6 - C_6^{-1} + C_3 + C_3^{-1} - C_2 + \sigma_a + \sigma_b + \sigma_c - \sigma_1 - \sigma_2 - \sigma_3\right) \tag{4}$$

$$P^{(4)} = \frac{1}{12}\left(e - C_6 - C_6^{-1} + C_3 + C_3^{-1} - C_2 - \sigma_a - \sigma_b - \sigma_c + \sigma_1 + \sigma_2 + \sigma_3\right) \tag{5}$$

$$P^{(5)} = \frac{1}{6}\left(2e + C_6 + C_6^{-1} - C_3 - C_3^{-1} - 2C_2\right) \tag{6}$$

$$P^{(6)} = \frac{1}{6}\left(2e - C_6 - C_6^{-1} - C_3 - C_3^{-1} + 2C_2\right) \tag{7}$$

With the aid of the multiplication table of symmetry group C_{6v} (Table 1), it is easy to show that $P^{(i)}P^{(i)} = P^{(i)}$ for $i = \{1, 2, \ldots, 6\}$. Furthermore, the orthogonality property also holds, i.e. $P^{(i)}P^{(j)} = 0$ if $i \neq j$.

In general, taking the idempotent $P^{(i)}$ corresponding to the irreducible representation $\Gamma^{(i)}$ (and associated with the subspace $S^{(i)}$), and applying this to each of the n freedoms $\phi_1, \phi_2, \ldots, \phi_n$ of an n-dimensional vibration problem, we obtain n symmetry-adapted freedoms, of which r_i (where $r_i \ll n$) are independent. The r_i independent symmetry-adapted freedoms may be taken as the basis vectors of subspace $S^{(i)}$. Subspace $S^{(i)}$ is therefore of dimension r_i. The sum of the dimensions of all k subspaces is equal to n: that is, $r_1 + r_2 + \ldots + r_k = n$ [1–3,28].

For any 1-dimensional irreducible representation $\Gamma^{(i)}$ of a symmetry group, the dimension r_i of the associated subspace $S^{(i)}$ is the smallest possible (i.e. no further decomposition of subspace $S^{(i)}$ is possible). However, for an m-dimensional irreducible representation (where m can be 2, 3, 4 or 5), the decomposition yielded by the application of idempotent $P^{(i)}$ results in a subspace $S^{(i)}$ that can still be decomposed further. Such *degenerate* subspaces are associated with repeating solutions (which, in the case of eigenvalue vibration problems, are repeating natural frequencies); the degree of repetition is equal to m. Irreducible representations of dimension 1 or 2 are typically associated with structural configurations belonging to cyclic (C) and dihedral (D) symmetry groups, whereas those of dimension greater than 2 are only encountered in the analysis of tetrahedral (T), octahedral (O) and icosahedral (I) configurations.

By reference to Table 3, and as already pointed out, vibration problems involving the symmetry group C_{6v} will have four normal subspaces $S^{(1)}$ to $S^{(4)}$ corresponding to the 1-dimensional irreducible representations $\Gamma^{(1)}$ to $\Gamma^{(4)}$ of the symmetry group, and two degenerate subspaces $S^{(5)}$ and $S^{(6)}$ corresponding to the 2-dimensional irreducible representations $\Gamma^{(5)}$ and $\Gamma^{(6)}$ of the symmetry group. As already defined, a degenerate subspace, in the present context, is one with eigenvalues that repeat two or more times within the subspace.

Idempotents $P^{(i)}$, when applied to the variables of a problem, generate the symmetry-adapted variables of the corresponding subspaces $S^{(i)}$. Since they contain repeating solutions, degenerate subspaces have the potential for further decomposition. However, for a 2-dimensional irreducible representation $\Gamma^{(i)}$, Eq. (1) only yields one idempotent $P^{(i)}$, which can only generate the symmetry-adapted variables for the parent (degenerate) subspace $S^{(i)}$, and is unable to further subdivide subspace $S^{(i)}$ into two smaller subspaces, thus separating the repeating solutions. If full decomposition of a degenerate subspace is required, alternative operators need to be found.

If full decomposition of a degenerate subspace can be achieved, this would allow the repeating solutions of the subspace to be computed within smaller subspaces, not only bypassing the numerical difficulties of computing coincident solutions, but also reducing computational effort by reducing the size of the problem. Algebraic operators in the form of simple linear combinations of the symmetry elements of the group, with the property of decomposing the degenerate subspace of problems belonging to the symmetry group

C_{4v} (this symmetry group is associated with physical configurations having the same symmetry as a square), were first proposed by the author in a study of the vibration modes of symmetric layered space grids [2], and subsequently applied to the eigenvalue vibration analysis of plates [15] and plane grids [28], and more recently, to the eigenvalue buckling analysis of plane frames [21,35].

More general formulations for deriving projection operators of the subspaces of symmetry groups have been in existence for a long time [6,31], but these formulations require the use of matrices of irreducible representations of the symmetry group. A formula for projection operators that can further decompose a degenerate subspace appears in the appendix of a paper by Healey & Treacy [32], but again implementation of that formulation requires matrices of the irreducible representations to be written down first. For the purposes of decomposing the two degenerate subspaces $S^{(5)}$ and $S^{(6)}$ of structural problems belonging to the symmetry group C_{6v}, it appears that explicit algebraic operators in the form of simple linear combinations of symmetry elements of group C_{6v}, similar to those already developed for the symmetry group C_{4v} [2] (and not requiring matrices of the irreducible representations of the symmetry group), are currently not available in the literature, to the best knowledge of the author. From a practical point of view, such simple operators would not only simplify the computation of doubly-repeating eigenvalues within each degenerate subspace, but also enable separation of the tangled modes of the degenerate subspace into two distinct symmetry types, thus affording a better understanding of vibration phenomena.

In the next section, we will present two pairs of operators for the automatic decomposition of the degenerate subspaces of problems specifically belonging to the symmetry group C_{6v}. Unlike more general existing approaches [6,31,32], these operators are not matrices, but are algebraic expressions comprising *simple linear combinations of the symmetry elements of the group* C_{6v}, and are therefore much easier to implement in practical engineering computations. We will illustrate their application by reference to the rectilinear vibrations of a spring-mass system, a problem which was studied in earlier work [27], but without the benefit of the present operators. Results for eigenvalues, as obtained using the proposed operators, will be compared with those previously obtained without the use of the operators, thus demonstrating the validity of the operators. In a further section of the paper, the small transverse vibrations of a hexagonal plane structural grid will be considered, and group theory applied to decompose the full space of the problem into 8 independent subspaces, four of which are the result of the full decomposition of the degenerate subspaces $S^{(5)}$ and $S^{(6)}$ using the proposed operators.

3. Operators for the full decomposition of degenerate subspaces of symmetry group C_{6v}

To simplify the computation of frequencies and modes in the degenerate subspaces $S^{(5)}$ and $S^{(6)}$ of eigenvalue vibration problems involving symmetry group C_{6v}, we seek two pairs of operators $\{P^{(i,1)};\ P^{(i,2)}\}$ (where $i = 5$ or $i = 6$) that are able to automatically subdivide the subspace $S^{(i)}$ ($i = \{5,\ 6\}$) into two smaller subspaces $\left\{S^{(i,1)};\ S^{(i,2)}\right\}$ spanned by linear combinations of the basis vectors of the parent subspace $S^{(i)}$, such that the basis vectors of the semi-subspace $S^{(i,1)}$ are orthogonal to those of the semi-subspace $S^{(i,2)}$. The orthogonality of semi-subspaces $S^{(i,1)}$ and $S^{(i,2)}$ would then allow them to be treated separately. We require these operators to satisfy the following four conditions:

$$P^{(i,1)} + P^{(i,2)} = P^{(i)} \tag{8}$$

$$P^{(i,1)}P^{(i,1)} = P^{(i,1)} \tag{9a}$$

$$P^{(i,2)}P^{(i,2)} = P^{(i,2)} \tag{9b}$$

$$P^{(i,1)}P^{(i,2)} = 0 \tag{10}$$

where $i = 5$ for subspace $S^{(5)}$ and $i = 6$ for subspace $S^{(6)}$. The first condition is the requirement that, for a given i, the sum of the two operators must equal the idempotent $P^{(i)}$ as given by Eq. (6) for subspace $S^{(5)}$ or Eq. (7) for subspace $S^{(6)}$. The second and third conditions require the two operators to have the property $P^{(j)}P^{(j)} = P^{(j)}$ common to all idempotents. The last condition requires that, for a given i, the two operators $P^{(i,1)}$ and $P^{(i,2)}$ must be orthogonal to each other, to ensure the orthogonality of the basis-vector sets of subspaces $S^{(i,1)}$ and $S^{(i,2)}$.

Let us first consider the degenerate subspace $S^{(5)}$. To preserve the rotational symmetries of idempotent $P^{(5)}$ (see Eq. (6)), let each of the sought operators $P^{(5,1)}$ and $P^{(5,2)}$ comprise half of $P^{(5)}$ and a linear combination of reflection elements $\{\sigma_a,\ \sigma_b,\ \sigma_c,\ \sigma_1,\ \sigma_2,\ \sigma_3\}$ that is of equal magnitude but of opposite sign, as follows:

$$P^{(5,1)} = \frac{1}{12}\left(2e + C_6 + C_6^{-1} - C_3 - C_3^{-1} - 2C_2 - 2\sigma_a + \sigma_b + \sigma_c - \sigma_1 - \sigma_2 + 2\sigma_3\right) \tag{11a}$$

$$P^{(5,2)} = \frac{1}{12}\left(2e + C_6 + C_6^{-1} - C_3 - C_3^{-1} - 2C_2 + 2\sigma_a - \sigma_b - \sigma_c + \sigma_1 + \sigma_2 - 2\sigma_3\right) \tag{11b}$$

These expressions automatically satisfy condition (8):

$$P^{(5,1)} + P^{(5,2)} = \frac{1}{6}\left(2e + C_6 + C_6^{-1} - C_3 - C_3^{-1} - 2C_2\right) = P^{(5)} \tag{12}$$

To check if $P^{(5,1)}$ and $P^{(5,2)}$ are orthogonal to each other (i.e. if condition (10) is satisfied), let

$$\overline{P}^{(5,1)} = 12P^{(5,1)};\ \overline{P}^{(5,2)} = 12P^{(5,2)} \tag{13ab}$$

Clearly, the orthogonality condition $P^{(5,1)}P^{(5,2)} = 0$ is satisfied if we can show that $\overline{P}^{(5,1)}\overline{P}^{(5,2)} = 0$. Introducing the parameters

$$a = 2e + C_6 + C_6^{-1} - C_3 - C_3^{-1} - 2C_2 \tag{14a}$$

$$b = 2\sigma_a - \sigma_b - \sigma_c + \sigma_1 + \sigma_2 - 2\sigma_3 \tag{14b}$$

we may re-write $\overline{P}^{(5,1)}$ and $\overline{P}^{(5,2)}$ as follows

$$\overline{P}^{(5,1)} = a - b;\ \overline{P}^{(5,2)} = a + b \tag{15ab}$$

Hence

$$\overline{P}^{(5,1)}\overline{P}^{(5,2)} = (a-b)(a+b) = a^2 - b^2 \tag{16}$$

Evaluating a^2 and b^2 with the aid of the multiplication table for group C_{6v} (Table 1), we obtain the following results, where in the tabulated part, the six terms of row i ($i = 1,\ 2,\ \ldots,\ 6$) are generated by multiplying term i of the first bracket by all six terms of the second bracket:

$a^2 = (2e + C_6 + C_6^{-1} - C_3 - C_3^{-1} - 2C_2)(2e + C_6 + C_6^{-1} - C_3 - C_3^{-1} - 2C_2) =$ (17a)

$+4e$	$+2C_6$	$+2C_6^{-1}$	$-2C_3$	$-2C_3^{-1}$	$-4C_2$
$+2C_6$	$+C_3$	$+e$	$-C_2$	$-C_6^{-1}$	$-2C_3^{-1}$
$+2C_6^{-1}$	$+e$	$+C_3^{-1}$	$-C_6$	$-C_2$	$-2C_3$
$-2C_3$	$-C_2$	$-C_6$	$+C_3^{-1}$	$+e$	$+2C_6^{-1}$
$-2C_3^{-1}$	$-C_6^{-1}$	$-C_2$	$+e$	$+C_3$	$+2C_6$
$-4C_2$	$-2C_3^{-1}$	$-2C_3$	$+2C_6^{-1}$	$+2C_6$	$+4e$

$= 12e + 6C_6 + 6C_6^{-1} - 6C_3 - 6C_3^{-1} - 12C_2$

$b^2 = (2\sigma_a - \sigma_b - \sigma_c + \sigma_1 + \sigma_2 - 2\sigma_3)(2\sigma_a - \sigma_b - \sigma_c + \sigma_1 + \sigma_2 - 2\sigma_3) =$ (17b)

$+4e$	$-2C_3$	$-2C_3^{-1}$	$+2C_6^{-1}$	$+2C_6$	$-4C_2$
$-2C_3^{-1}$	$+e$	$+C_3$	$-C_2$	$-C_6^{-1}$	$+2C_6$
$-2C_3$	$+C_3^{-1}$	$+e$	$-C_6$	$-C_2$	$+2C_6^{-1}$
$+2C_6$	$-C_2$	$-C_6^{-1}$	$+e$	$+C_3$	$-2C_3^{-1}$
$+2C_6^{-1}$	$-C_6$	$-C_2$	$+C_3^{-1}$	$+e$	$-2C_3$
$-4C_2$	$+2C_6^{-1}$	$+2C_6$	$-2C_3$	$-2C_3^{-1}$	$+4e$

$= 12e - 6C_3 - 6C_3^{-1} + 6C_6^{-1} + 6C_6 - 12C_2$

Clearly, $a^2 = b^2$, hence

$$\overline{P}^{(5,1)}\overline{P}^{(5,2)} = a^2 - b^2 = 0 \tag{18}$$

and so

$$P^{(5,1)}P^{(5,2)} = 0 \tag{19}$$

proving the orthogonality of $P^{(5,1)}$ and $P^{(5,2)}$.

Similarly, for the degenerate subspace $S^{(6)}$, in order to preserve the rotational symmetries of idempotent $P^{(6)}$ (see Eq. (7)), we assume the decomposition operators $P^{(6,1)}$ and $P^{(6,2)}$ to comprise half of $P^{(6)}$ and an equal but opposite linear combination of reflection elements $\{\sigma_a,\ \sigma_b,\ \sigma_c,\ \sigma_1,\ \sigma_2,\ \sigma_3\}$, as follows:

$$P^{(6,1)} = \frac{1}{12}\left(2e - C_6 - C_6^{-1} - C_3 - C_3^{-1} + 2C_2 + 2\sigma_a - \sigma_b - \sigma_c - \sigma_1 - \sigma_2 + 2\sigma_3\right) \tag{20a}$$

$$P^{(6,2)} = \frac{1}{12}\left(2e - C_6 - C_6^{-1} - C_3 - C_3^{-1} + 2C_2 - 2\sigma_a + \sigma_b + \sigma_c + \sigma_1 + \sigma_2 - 2\sigma_3\right) \tag{20b}$$

The above expressions automatically satisfy condition (8):

$$P^{(6,1)} + P^{(6,2)} = \frac{1}{6}\left(2e - C_6 - C_6^{-1} - C_3 - C_3^{-1} + 2C_2\right) = P^{(6)} \tag{21}$$

To check if $P^{(6,1)}$ and $P^{(6,2)}$ are orthogonal to each other (i.e. if condition (10) is satisfied), let

$$\overline{P}^{(6,1)} = 12P^{(6,1)};\ \overline{P}^{(6,2)} = 12P^{(6,2)} \tag{22ab}$$

Clearly, the orthogonality condition $P^{(6,1)}P^{(6,2)} = 0$ is fulfilled if we can show that $\overline{P}^{(6,1)}\overline{P}^{(6,2)} = 0$. If we define parameters c and s as follows:-

$$c = 2e - C_6 - C_6^{-1} - C_3 - C_3^{-1} + 2C_2 \tag{23a}$$

$$s = 2\sigma_a - \sigma_b - \sigma_c - \sigma_1 - \sigma_2 + 2\sigma_3 \tag{23b}$$

then $\overline{P}^{(6,1)}$ and $\overline{P}^{(6,2)}$ may be re-written as follows

$$\overline{P}^{(6,1)} = c + s;\ \overline{P}^{(6,2)} = c - s \tag{24ab}$$

Hence

$$\overline{P}^{(6,1)}\overline{P}^{(6,2)} = (c+s)(c-s) = c^2 - s^2 \tag{25}$$

Evaluating c^2 and s^2 with the aid of the multiplication table for group C_{6v} (Table 1), we obtain the following results, where in the tabulated part, the six terms of row i ($i = 1,\ 2,\ \ldots,\ 6$) are generated by multiplying term i of the first bracket by all six terms of the second bracket:

$c^2 = (2e - C_6 - C_6^{-1} - C_3 - C_3^{-1} + 2C_2)(2e - C_6 - C_6^{-1} - C_3 - C_3^{-1}z + 2C_2) =$

$+4e$	$-2C_6$	$-2C_6^{-1}$	$-2C_3$	$-2C_3^{-1}$	$+4C_2$
$-2C_6$	$+C_3$	$+e$	$+C_2$	$+C_6^{-1}$	$-2C_3^{-1}$
$-2C_6^{-1}$	$+e$	$+C_3^{-1}$	$+C_6$	$+C_2$	$-2C_3$
$-2C_3$	$+C_2$	$+C_6$	$+C_3^{-1}$	$+e$	$-2C_6^{-1}$
$-2C_3^{-1}$	$+C_6^{-1}$	$+C_2$	$+e$	$+C_3$	$-2C_6$
$+4C_2$	$-2C_3^{-1}$	$-2C_3$	$-2C_6^{-1}$	$-2C_6$	$+4e$

(26a)

$= 12e - 6C_6 - 6C_6^{-1} - 6C_3 - 6C_3^{-1} + 12C_2$

$s^2 = (2\sigma_a - \sigma_b - \sigma_c - \sigma_1 - \sigma_2 + 2\sigma_3)(2\sigma_a - \sigma_b - \sigma_c - \sigma_1 - \sigma_2 + 2\sigma_3) =$

$+4e$	$-2C_3$	$-2C_3^{-1}$	$-2C_6^{-1}$	$-2C_6$	$+4C_2$
$-2C_3^{-1}$	$+e$	$+C_3$	$+C_2$	$+C_6^{-1}$	$-2C_6$
$-2C_3$	$+C_3^{-1}$	$+e$	$+C_6$	$+C_2$	$-2C_6^{-1}$
$-2C_6$	$+C_2$	$+C_6^{-1}$	$+e$	$+C_3$	$-2C_3^{-1}$
$-2C_6^{-1}$	$+C_6$	$+C_2$	$+C_3^{-1}$	$+e$	$-2C_3$
$+4C_2$	$-2C_6^{-1}$	$-2C_6$	$-2C_3$	$-2C_3^{-1}$	$+4e$

(26b)

$= 12e - 6C_3 - 6C_3^{-1} - 6C_6^{-1} - 6C_6 + 12C_2$

Clearly, $c^2 = s^2$, hence

$$\overline{P}^{(6,1)}\overline{P}^{(6,2)} = c^2 - s^2 = 0 \tag{27}$$

and so

$$P^{(6,1)}P^{(6,2)} = 0 \tag{28}$$

proving the orthogonality of $P^{(6,1)}$ and $P^{(6,2)}$.

To check if $P^{(5,1)}$ satisfies condition (9a), consider the square of $\overline{P}^{(5,1)}$:-

$$\overline{P}^{(5,1)}\overline{P}^{(5,1)} = \left(2e + C_6 + C_6^{-1} - C_3 - C_3^{-1} - 2C_2 - 2\sigma_a + \sigma_b + \sigma_c - \sigma_1 - \sigma_2 + 2\sigma_3\right)$$

$$\times\left(2e + C_6 + C_6^{-1} - C_3 - C_3^{-1} - 2C_2 - 2\sigma_a + \sigma_b + \sigma_c - \sigma_1 - \sigma_2 + 2\sigma_3\right) \tag{29}$$

With the aid of the multiplication table for group C_{6v} (Table 1), we may write down the 144 results of the multiplication on the right-hand side of Eq. (29) in tabular format (where the 12 terms of row i ($i = 1,\ 2,\ \ldots,\ 12$) of the tabular format are generated by multiplying term i of the first bracketed expression on the right-hand side of Eq. (29) by all 12 terms of the second bracketed expression), and then simplify the ensuing 144-term expression, as follows:

$$\begin{array}{cccccccccccc}
+4e & +2C_6 & +2C_6^{-1} & -2C_3 & -2C_3^{-1} & -4C_2 & -4\sigma_a & +2\sigma_b & +2\sigma_c & -2\sigma_1 & -2\sigma_2 & +4\sigma_3 \\
+2C_6 & +C_3 & +e & -C_2 & -C_6^{-1} & -2C_3^{-1} & -2\sigma_1 & +\sigma_2 & +\sigma_3 & -\sigma_c & -\sigma_a & +2\sigma_b \\
+2C_6^{-1} & +e & +C_3^{-1} & -C_6 & -C_2 & -2C_3 & -2\sigma_2 & +\sigma_3 & +\sigma_1 & -\sigma_a & -\sigma_b & +2\sigma_c \\
-2C_3 & -C_2 & -C_6 & +C_3^{-1} & +e & +2C_6^{-1} & +2\sigma_c & -\sigma_a & -\sigma_b & +\sigma_3 & +\sigma_1 & -2\sigma_2 \\
-2C_3^{-1} & -C_6^{-1} & -C_2 & +e & +C_3 & +2C_6 & +2\sigma_b & -\sigma_c & -\sigma_a & +\sigma_2 & +\sigma_3 & -2\sigma_1 \\
-4C_2 & -2C_3^{-1} & -2C_3 & +2C_6^{-1} & +2C_6 & +4e & +4\sigma_3 & -2\sigma_1 & -2\sigma_2 & +2\sigma_b & +2\sigma_c & -4\sigma_a \\
-4\sigma_a & -2\sigma_2 & -2\sigma_1 & +2\sigma_b & +2\sigma_c & +4\sigma_3 & +4e & -2C_3 & -2C_3^{-1} & +2C_6^{-1} & +2C_6 & -4C_2 \\
+2\sigma_b & +\sigma_3 & +\sigma_2 & -\sigma_c & -\sigma_a & -2\sigma_1 & -2C_3^{-1} & +e & +C_3 & -C_2 & -C_6^{-1} & +2C_6 \\
+2\sigma_c & +\sigma_1 & +\sigma_3 & -\sigma_a & -\sigma_b & -2\sigma_2 & -2C_3 & +C_3^{-1} & +e & -C_6 & -C_2 & +2C_6^{-1} \\
-2\sigma_1 & -\sigma_a & -\sigma_c & +\sigma_2 & +\sigma_3 & +2\sigma_b & +2C_6 & -C_2 & -C_6^{-1} & +e & +C_3 & -2C_3^{-1} \\
-2\sigma_2 & -\sigma_b & -\sigma_a & +\sigma_3 & +\sigma_1 & +2\sigma_c & +2C_6^{-1} & -C_6 & -C_2 & +C_3^{-1} & +e & -2C_3 \\
+4\sigma_3 & +2\sigma_c & +2\sigma_b & -2\sigma_1 & -2\sigma_2 & -4\sigma_a & -4C_2 & +2C_6^{-1} & +2C_6 & -2C_3 & -2C_3^{-1} & +4e
\end{array} \tag{30}$$

$$= 24e + 12C_6 + 12C_6^{-1} - 12C_3 - 12C_3^{-1} - 24C_2$$
$$\quad -24\sigma_a + 12\sigma_b + 12\sigma_c - 12\sigma_1 - 12\sigma_2 + 24\sigma_3$$

$$= 12\{2e + C_6 + C_6^{-1} - C_3 - C_3^{-1} - 2C_2 - 2\sigma_a + \sigma_b + \sigma_c - \sigma_1 - \sigma_2 + 2\sigma_3\}$$

$$= 12\left\{12P^{(5,1)}\right\}$$

Thus, making use of the final result in Eq. (30), the square of $\overline{P}^{(5,1)}$ becomes

$$\overline{P}^{(5,1)}\overline{P}^{(5,1)} = 12\left\{12P^{(5,1)}\right\} \tag{31}$$

Using Eq. (13a) to eliminate $\overline{P}^{(5,1)}$ from the left-hand side of Eq. (31), we obtain

$$\left\{12P^{(5,1)}\right\}\left\{12P^{(5,1)}\right\} = 12\left\{12P^{(5,1)}\right\} \tag{32}$$

which simplifies to

$$P^{(5,1)}P^{(5,1)} = P^{(5,1)} \tag{33}$$

showing that $P^{(5,1)}$ has the property of an idempotent.

Similarly, by expanding the square of $\overline{P}^{(5,2)}$, it can be shown that

$$P^{(5,2)}P^{(5,2)} = P^{(5,2)} \tag{34}$$

and by expanding the squares of $\overline{P}^{(6,1)}$ and $\overline{P}^{(6,2)}$, the following results can be established:

$$P^{(6,1)}P^{(6,1)} = P^{(6,1)} \tag{35}$$

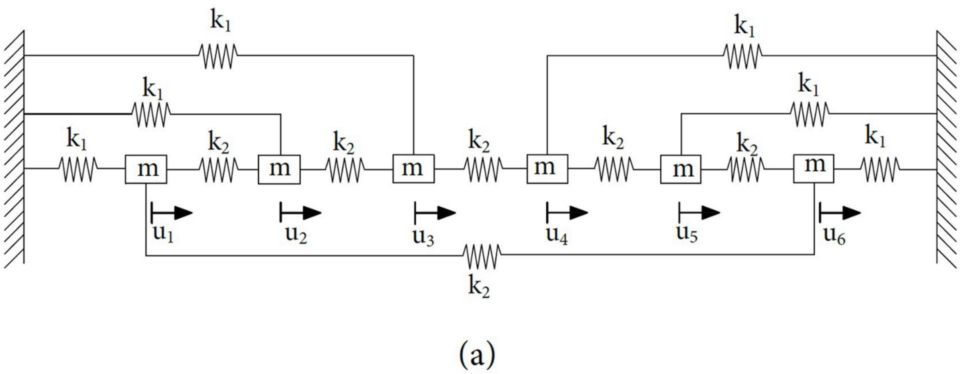
$$P^{(6,2)}P^{(6,2)} = P^{(6,2)} \tag{36}$$

Thus, both pairs of operators $\left\{P^{(5,1)};\ P^{(5,2)}\right\}$ and $\left\{P^{(6,1)};\ P^{(6,2)}\right\}$ satisfy all four conditions given by Eqs. (8)–(10). They are effectively the idempotents of subspaces $\left\{S^{(5,1)};\ S^{(5,2)}\right\}$ and $\left\{S^{(6,1)};\ S^{(6,2)}\right\}$. When applied upon the normal freedoms of a vibration problem with C_{6v} symmetry, these operators will automatically generate orthogonal sets of symmetry-adapted freedoms, which is equivalent to decomposing the degenerate parent subspaces $S^{(5)}$ and $S^{(6)}$. If the parent subspace $S^{(i)}$ (where $i = 5$ or $i = 6$) is of dimension r_i (this is always an even integer), the operators $P^{(i,1)}$ and $P^{(i,2)}$ automatically generate the $r_i/2$ basis vectors of subspace $S^{(i,1)}$ and the $r_i/2$ basis vectors of subspace $S^{(i,2)}$, respectively, thus decomposing subspace $S^{(i)}$ into two subspaces that are each half the size of subspace $S^{(i)}$.

The operators given by Eqs. (11) and (20) are quite general, being applicable to any problem that has C_{6v} symmetry. Thus, in any structural or mechanical problem involving the symmetry group C_{6v}, projection operators for subspaces $S^{(5,1)}$, $S^{(5,2)}$, $S^{(6,1)}$ and $S^{(6,2)}$ need not be sought anymore; they are given by expressions (11) and (20). Once validated, expressions (11) and (20) may be regarded as general computational formulae for the automatic decomposition of subspaces $S^{(5)}$ and $S^{(6)}$ of any structural or mechanical problem belonging to the symmetry group C_{6v}, whether the problem concerns the vibration, stability, static or kinematic behaviour of the system. Being simple algebraic expressions (linear combinations of the symmetry elements of the group), they are very easy to use (a major advantage from a practical engineering point of view), and will automatically decompose the associated degenerate subspace. In the next section, we will illustrate the application of the operators by consideration of the rectilinear vibrations of a spring-mass dynamic system, at the same time validating them.

4. Application to the vibration of a spring-mass system and validation of operators

Fig. 3(a) shows a spring-mass dynamic model with 6 d.o.f $\{u_1,\ u_2, \ldots,\ u_6\}$ representing the rectilinear motions of 6 masses $\{m_1,\ m_2, \ldots, m_6\}$, each of magnitude m as shown. Each mass is connected to a rigid support by a spring of stiffness k_1, and to two other masses by



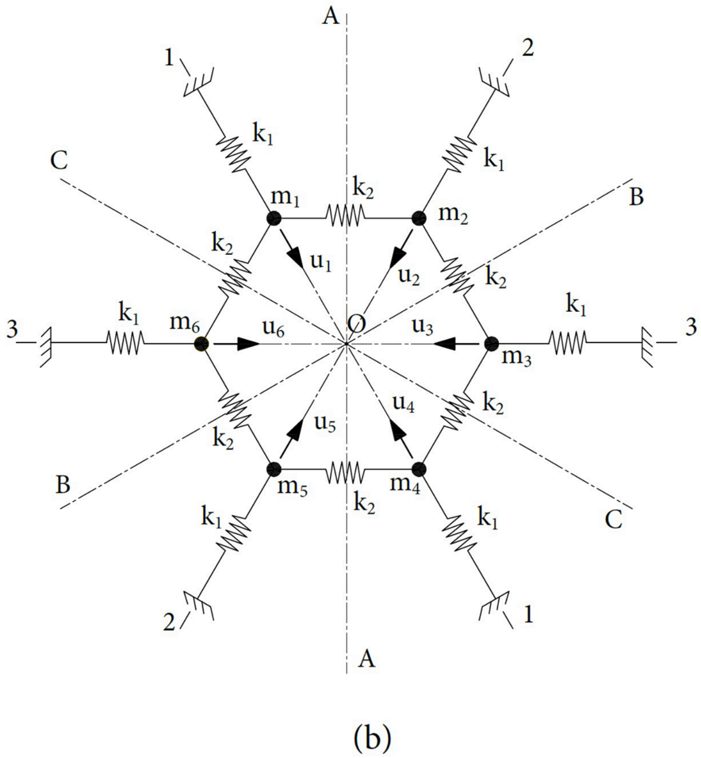


Fig. 3. A 6 d.o.f. spring–mass system: (a) actual configuration; (b) transformed configuration with C_{6v} symmetry [27].

springs of stiffness k_2, as shown in the diagram. The configuration has C_{6v} symmetry, which becomes clearer if it is re-drawn with the six symmetry axes labelled as shown in Fig. 3(b). The system in Fig. 3(a) is dynamically equivalent to that in Fig. 3(b), if the masses in Fig. 3(b) are assumed to be mounted on rails or rollers that permit movement along the axes $1-1$, $2-2$ and $3-3$ only. This example was considered in a previous study [27], where natural frequencies of vibration for all subspaces of the problem were computed, but without the benefit of the operators $\{P^{(5,1)};\ P^{(5,2)}\}$ and $\{P^{(6,1)};\ P^{(6,2)}\}$ that have been proposed in the present contribution.

In general, for eigenvalue vibration problems of this type, and assuming the system has n d.o.f, group-theoretic decomposition allows the n-dimensional problem to be analysed as a series of k smaller eigenvalue problems of dimensions r_i ($i = 1,\ 2,\ \ldots,\ k$), where $r_1 + r_2 + \ldots + r_k = n$, and each of these small problems is associated with a symmetry subspace of the original vector space of the problem [1–3,28]. Compared with conventional eigenvalue vibration analysis, the group-theoretic approach has the advantage of significantly reducing overall computation effort; it also allows the symmetry properties of the vibration modes to be understood more fully. In the case of degenerate subspaces with repeating eigenvalues, the advantage of the proposed operators (Eqs. (11) and (20)) lies in allowing a further decomposition of such subspaces to be achieved, which makes the computation of repeating eigenvalues even easier, at the same time separating the mixed modes of the degenerate subspaces into two very distinct symmetry categories. This example will illustrate these merits.

By applying idempotents $\{P^{(1)},\ P^{(2)},\ \ldots,\ P^{(6)}\}$, as given by Eqs. (2)–(7), on the freedoms $\{u_1,\ u_2, \ldots,\ u_6\}$ of the transformed configuration (Fig. 3(b)), it was found [27] that subspace $S^{(1)}$ is 1-dimensional (i.e. it has only one basis vector), subspaces $S^{(2)}$ and $S^{(3)}$ are null subspaces, subspace $S^{(4)}$ is 1-dimensional, and subspaces $S^{(5)}$ and $S^{(6)}$ are each 2-dimensional. Subspaces $S^{(5)}$ and $S^{(6)}$ were shown to each have doubly repeating natural frequencies, but these frequencies were computed using the basis vectors of the 2-dimensional subspaces. Here, we will use the operator pairs $\{P^{(5,1)};\ P^{(5,2)}\}$ and $\{P^{(6,1)};\ P^{(6,2)}\}$ as given by Eqs. (11) and (20), to illustrate their application to vibration problems, and also to check if the same results (as obtained in previous work [27]) can be obtained, thus validating the operators.

4.1. Basis vectors of the semi-subspaces of the degenerate subspaces $S^{(5)}$ and $S^{(6)}$

Let us consider the degenerate subspace $S^{(5)}$ first. Applying the operator $P^{(5,1)}$ (refer to Eq. (11a)) to the first three freedoms $\{u_1,\ u_2,\ u_3\}$ in Fig. 3(b) in turn, we obtain the first three symmetry-adapted freedoms of semi-subspace $S^{(5,1)}$ as follows:

$$P^{(5,1)}u_1 = \frac{1}{12}\left(2e + C_6 + C_6^{-1} - C_3 - C_3^{-1} - 2C_2 - 2\sigma_a + \sigma_b + \sigma_c - \sigma_1 - \sigma_2 + 2\sigma_3\right)u_1$$

$$= \frac{1}{12}(2u_1 + u_2 + u_6 - u_3 - u_5 - 2u_4 - 2u_2 + u_4 + u_6 - u_1 - u_3 + 2u_5) = \frac{1}{12}(u_1 - u_2 - 2u_3 - u_4 + u_5 + 2u_6) \tag{37a}$$

$$P^{(5,1)}u_2 = \frac{1}{12}\left(2e + C_6 + C_6^{-1} - C_3 - C_3^{-1} - 2C_2 - 2\sigma_a + \sigma_b + \sigma_c - \sigma_1 - \sigma_2 + 2\sigma_3\right)u_2$$

$$= \frac{1}{12}(2u_2 + u_3 + u_1 - u_4 - u_6 - 2u_5 - 2u_1 + u_3 + u_5 - u_6 - u_2 + 2u_4) = \frac{1}{12}(-u_1 + u_2 + 2u_3 + u_4 - u_5 - 2u_6) = -P^{(5,1)}u_1 \tag{37b}$$

$$P^{(5,1)}u_3 = \frac{1}{12}\left(2e + C_6 + C_6^{-1} - C_3 - C_3^{-1} - 2C_2 - 2\sigma_a + \sigma_b + \sigma_c - \sigma_1 - \sigma_2 + 2\sigma_3\right)u_3$$

$$= \frac{1}{12}(2u_3 + u_4 + u_2 - u_5 - u_1 - 2u_6 - 2u_6 + u_2 + u_4 - u_5 - u_1 + 2u_3) = \frac{2}{12}(-u_1 + u_2 + 2u_3 + u_4 - u_5 - 2u_6) = -2P^{(5,1)}u_1 \tag{37c}$$

Applying the operator $P^{(5,1)}$ to the remaining three freedoms $\{u_4,\ u_5,\ u_6\}$ in the same fashion, we find that the ensuing symmetry-adapted freedoms are all linearly related to $P^{(5,1)}u_1$, so there is only one independent symmetry-adapted freedom. The basis vector of semi-subspace $S^{(5,1)}$ may be written down from $P^{(5,1)}u_1$ by ignoring the scalar $1/12$, as follows:

$$\begin{aligned}\Phi^{(5,1)} &= u_1 - u_2 - 2u_3 - u_4 + u_5 + 2u_6 \\ &= \{+1 \quad -1 \quad -2 \quad -1 \quad +1 \quad +2\}\{u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6\}^T \\ &= \{B^{(5,1)}\}^T\{U\}\end{aligned} \tag{38}$$

where the *column* vectors $\{B^{(5,1)}\}$ and $\{U\}$ are given by

$$\{B^{(5,1)}\} = \{+1 \quad -1 \quad -2 \quad -1 \quad +1 \quad +2\}^T \tag{39}$$

$$\{U\} = \{u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6\}^T \tag{40}$$

Applying operator $P^{(5,2)}$ (see Eq. (11b)) to the six freedoms $\{u_1,\ u_2,\ \ldots,\ u_6\}$ in a similar way, we find that the ensuing symmetry-adapted freedoms are all linearly related to $P^{(5,2)}u_1$, implying that there is only one independent symmetry-adapted freedom. The basis vector of semi-subspace $S^{(5,2)}$ may be written down from $P^{(5,2)}u_1$ by ignoring the scalar 3/12, as follows:

$$\Phi^{(5,2)} = u_1 + u_2 - u_4 - u_5$$

$$= \{+1 \quad +1 \quad 0 \quad -1 \quad -1 \quad 0 \quad u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6\}^T$$

$$= \left\{B^{(5,2)}\right\}^T \{U\} \tag{41}$$

where the column vector $\left\{B^{(5,2)}\right\}$ is given by

$$\left\{B^{(5,2)}\right\} = \{+1 \quad +1 \quad 0 \quad -1 \quad -1 \quad 0\}^T \tag{42}$$

and the column vector $\{U\}$ is as already defined (Eq. (40)).

By reference to Eqs. (39) and (42),

$$\left\{B^{(5,1)}\right\}^T \left\{B^{(5,2)}\right\} = \{+1 \quad -1 \quad -2 \quad -1 \quad +1 \quad +2 \quad +1 \quad +1 \quad 0 \quad -1 \quad -1 \quad 0\}^T$$

$$= +1 - 1 + 0 + 1 - 1 + 0 = 0 \tag{43}$$

showing that basis vectors $\Phi^{(5,1)}$ and $\Phi^{(5,2)}$, spanning the semi-subspaces $S^{(5,1)}$ and $S^{(5,2)}$ respectively, are orthogonal.

Plots of basis vectors $\Phi^{(5,1)}$ and $\Phi^{(5,2)}$ are shown in Fig. 4. They convey important qualitative information, since the symmetries of these plots are a prediction of the symmetries of the vibration modes of semi-subspaces $S^{(5,1)}$ and $S^{(5,2)}$, before a full eigenvalue analysis is performed.

Basis vector $\Phi^{(5,1)}$ is *symmetric* about axis $3-3$ and *antisymmetric* about axis $A-A$, while basis vector $\Phi^{(5,2)}$ is *antisymmetric* about axis $3-3$ and *symmetric* about axis $A-A$. Within the parent subspace $S^{(5)}$, vibration modes of these two symmetry types are mixed, but as a result of using the new operators $\left\{P^{(5,1)};\ P^{(5,2)}\right\}$, it has been possible to separate them out. This separation of the mixed symmetries of degenerate subspaces is yet another benefit of the proposed operators. The plots in Fig. 4 also illustrate the orthogonality of the basis vectors $\Phi^{(5,1)}$ and $\Phi^{(5,2)}$, noting that axes $A-A$ and $3-3$ are *perpendicular* to each other.

We proceed in the same way for the degenerate subspace $S^{(6)}$. Applying operators $P^{(6,1)}$ and $P^{(6,2)}$ (as given by Eqs. (20a) and (20b)) to the six freedoms $\{u_1,\ u_2,\ \ldots,\ u_6\}$, we find that semi-subspaces $S^{(6,1)}$ and $S^{(6,2)}$ are each 1-dimensional (i.e. each has only one independent symmetry-adapted freedom). Their basis vectors may be taken as follows:

$$\Phi^{(6,1)} = u_1 + u_2 - 2u_3 + u_4 + u_5 - 2u_6$$

$$= \{+1 \quad +1 \quad -2 \quad +1 \quad +1 \quad -2 \quad u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6\}^T$$

$$= \left\{B^{(6,1)}\right\}^T \{U\} \tag{44}$$

$$\Phi^{(6,2)} = u_1 - u_2 + u_4 - u_5$$

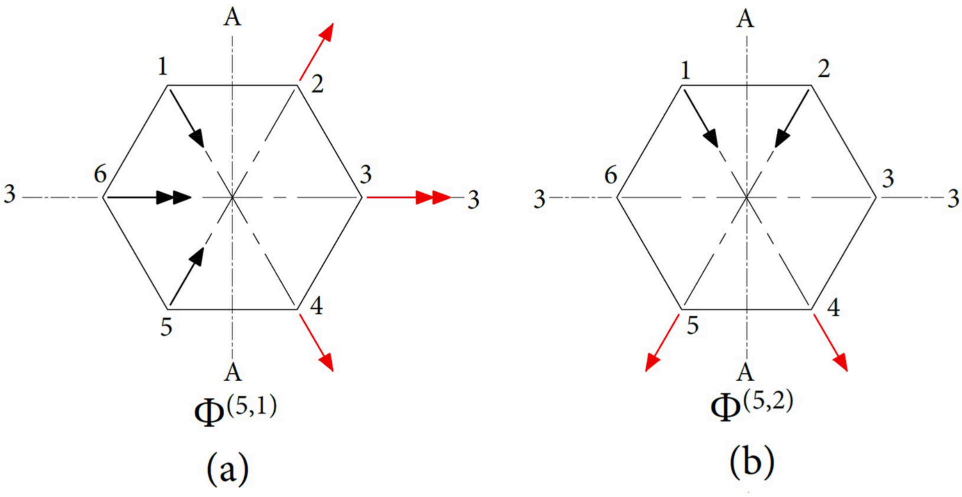


Fig. 4. Plots of basis vectors of semi-subspaces $S^{(5,1)}$ and $S^{(5,2)}$ of the spring-mass system. (For an interpretation of the references to colour in this figure, the reader is referred to the web version of this article.)

$$= \{+1 \quad -1 \quad 0 \quad +1 \quad -1 \quad 0 \quad u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6\}^T$$

$$= \{B^{(6,2)}\}^T\{U\} \tag{45}$$

where the column vectors $\{B^{(6,1)}\}$ and $\{B^{(6,2)}\}$ are given by

$$\{B^{(6,1)}\} = \{+1 \quad +1 \quad -2 \quad +1 \quad +1 \quad -2\}^T \tag{46}$$

$$\{B^{(6,2)}\} = \{+1 \quad -1 \quad 0 \quad +1 \quad -1 \quad 0\}^T \tag{47}$$

and the column vector $\{U\}$ is as already defined (Eq. (40)).

By reference to Eqs. (46) and (47),

$$\{B^{(6,1)}\}^T\{B^{(6,2)}\} = \{+1 \quad +1 \quad -2 \quad +1 \quad +1 \quad -2\}\{+1 \quad -1 \quad 0 \quad +1 \quad -1 \quad 0\}^T$$

$$= +1 - 1 + 0 + 1 - 1 + 0 = 0 \tag{48}$$

proving the orthogonality of basis vectors $\Phi^{(6,1)}$ and $\Phi^{(6,2)}$, which span the semi-subspaces $S^{(6,1)}$ and $S^{(6,2)}$ respectively.

Plots of basis vectors $\Phi^{(6,1)}$ and $\Phi^{(6,2)}$ are shown in Fig. 5. From these, we can see that basis vector $\Phi^{(6,1)}$ is *symmetric* about both the $A-A$ axis (vertical) and the $3-3$ axis (horizontal), while basis vector $\Phi^{(6,2)}$ is *antisymmetric* about both the $A-A$ and $3-3$ axes. These two symmetry types are characteristic of semi-subspaces $S^{(6,1)}$ and $S^{(6,2)}$, and would have co-existed within the parent subspace $S^{(6)}$, had it not been for the separation achieved by means of the new operators $P^{(6,1)}$ and $P^{(6,2)}$. The plots in Fig. 5 are a pictorial illustration of the orthogonality of basis vectors $\Phi^{(6,1)}$ and $\Phi^{(6,2)}$, noting that axes $A-A$ and $3-3$ are mutually perpendicular.

4.2. *Symmetry-adapted stiffnesses and natural frequencies*

To solve for the eigenvalues of the system within the independent subspaces $\{S^{(5,1)};\ S^{(5,2)}\}$ and $\{S^{(6,1)};\ S^{(6,2)}\}$, we return to the original physical system as shown in Fig. 3(a). Taking the basis vector $\Phi^{(5,1)}$ as given by Eq. (38), we assign unit values of $\{u_1,\ u_2,\ u_3,\ u_4,\ u_5,\ u_6\}$, then *simultaneously* apply the ensuing displacements on the masses $\{m_1,\ m_2,\ m_3,\ m_4,\ m_5,\ m_6\}$ respectively. The applied displacements $\{b_1,\ b_2,\ b_3,\ b_4,\ b_5,\ b_6\}$ on these masses are simply given by the elements of $B^{(5,1)}$ in Eq. (39). We do the same for basis vector $\Phi^{(5,2)}$ (Eq. (41)) using the $B^{(5,2)}$ values in Eq. (42), for basis vector $\Phi^{(6,1)}$ (Eq. (44)) using the $B^{(6,1)}$ values in Eq. (46), and for basis vector $\Phi^{(6,2)}$ (Eq. (45)) using the $B^{(6,2)}$ values in Eq. (47). These applications of the b displacements are illustrated in Fig. 6 for subspaces $S^{(5,1)}$ and $S^{(5,2)}$, and Fig. 7 for subspaces $S^{(6,1)}$ and $S^{(6,2)}$. Positive values of b point towards the right, and negative values towards the left.

For any lumped-parameter model of a symmetric structural or mechanical system undergoing linear vibrations, let us assume the space V of the problem has already been decomposed into a number of independent subspaces $S^{(\mu)}$ based on an appropriate symmetry group. In the case of systems with C_{6v} symmetry, and taking account of the further splitting of subspaces $S^{(5)}$ and $S^{(6)}$ via the two pairs of operators $\{P^{(5,1)};\ P^{(5,2)}\}$ and $\{P^{(6,1)};\ P^{(6,2)}\}$ as proposed in the present paper, there would be eight subspaces in total: $S^{(1)}$, $S^{(2)}$, $S^{(3)}$,

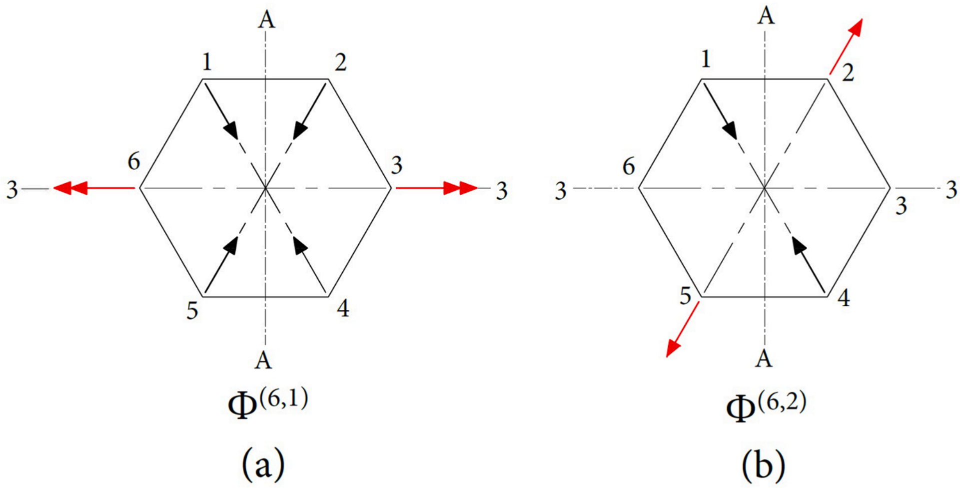


Fig. 5. Plots of basis vectors of semi-subspaces $S^{(6,1)}$ and $S^{(6,2)}$ of the spring-mass system. (For an interpretation of the references to colour in this figure, the reader is referred to the web version of this article.)

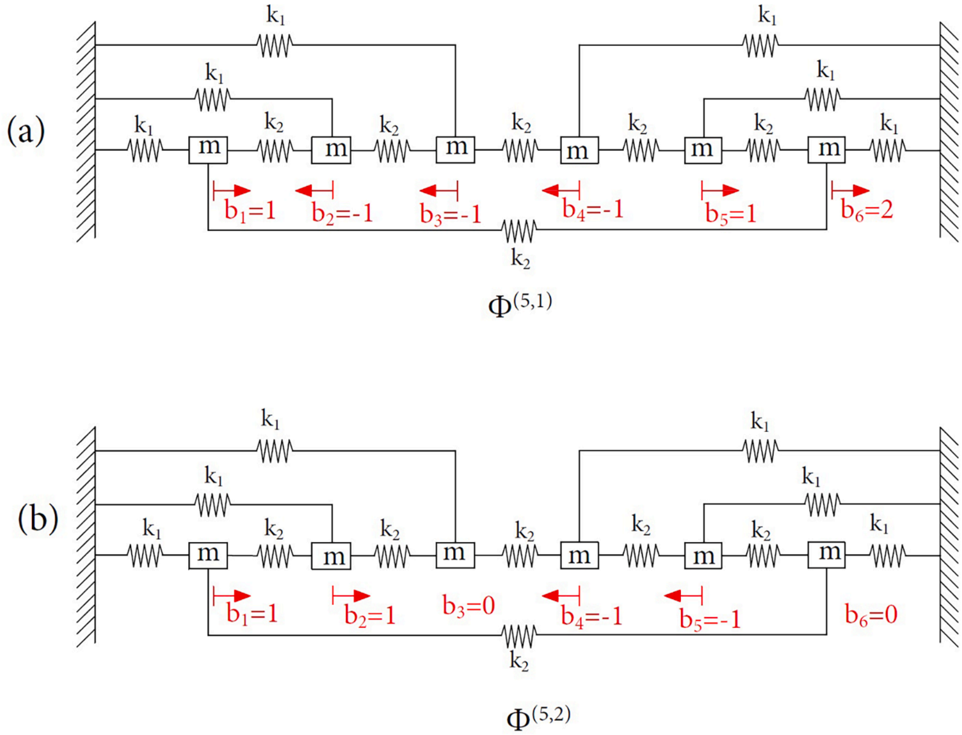


Fig. 6. Application of b displacements for the 6 d.o.f. spring-mass dynamic system: (a) displacements of subspace $S^{(5,1)}$; (b) displacements of subspace $S^{(5,2)}$. (For an interpretation of the references to colour in this figure, the reader is referred to the web version of this article.)

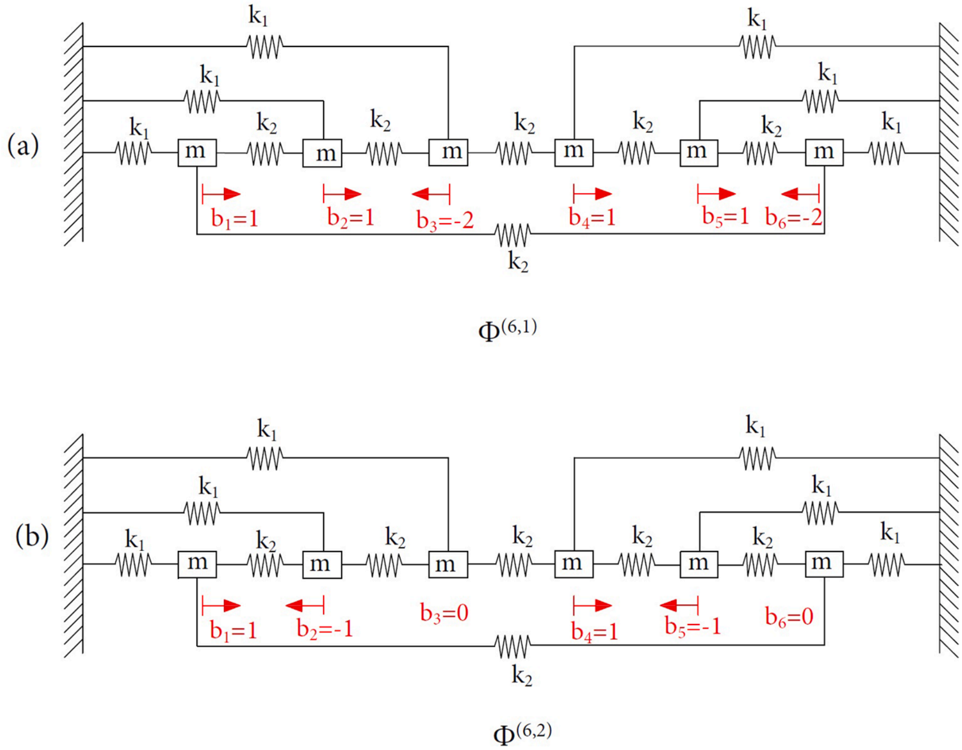


Fig. 7. Application of b displacements for the 6 d.o.f. spring-mass dynamic system: (a) displacements of subspace $S^{(6,1)}$; (b) displacements of subspace $S^{(6,2)}$. (For an interpretation of the references to colour in this figure, the reader is referred to the web version of this article.)

$S^{(4)}$, $S^{(5,1)}$, $S^{(5,2)}$, $S^{(6,1)}$ and $S^{(6,2)}$.

For a given subspace $S^{(\mu)}$ spanned by r basis vectors $\left\{\Phi_1^{(\mu)}, \Phi_2^{(\mu)}, \dots, \Phi_r^{(\mu)}\right\}$, we define the symmetry-adapted stiffness coefficient $\widetilde{k}_{ij}$ $(i = 1, 2, \dots, r; j = 1, 2, \dots, r)$ as the value of the restoring force at any *one* of the coordinates of Φ_i (it is normal to take the first coordinate of Φ_i) due to the simultaneous application of unit values of displacements at *all* of the coordinates of Φ_j. The assembly of all coefficients $\widetilde{k}_{ij}$ is the symmetry-adapted stiffness matrix $\widetilde{K}^{(\mu)}$ for subspace $S^{(\mu)}$.

In the present example of the 6 d.o.f. spring-mass system, all subspaces of present interest – $S^{(5,1)}$, $S^{(5,2)}$, $S^{(6,1)}$ and $S^{(6,2)}$ – are 1-dimensional (i.e. each has only one basis vector), so $\widetilde{K}^{(\mu)}$ will be a 1×1 matrix for all subspaces, comprising only one symmetry-adapted stiffness coefficient $\widetilde{k}_{1,1}^{(\mu)}$, which we will simply write as $\widetilde{k}^{(\mu)}$ (the subscripts i, j are not necessary). For each of these 1-dimensional subspaces $S^{(\mu)}$, the symmetry-adapted stiffness coefficient $\widetilde{k}^{(\mu)}$ is simply the net value of the restoring force at any of the masses of basis vector $\Phi^{(\mu)}$ when the appropriate set of b values (as shown in Figs. 6 and 7) is simultaneously applied on the system.

Tables 4 to 7 show the restoring spring forces ($F = kx$) on each of the six masses, resulting from the simultaneous application of the b displacements as shown in Figs. 6 and 7. For each mass, the entry in column 3 is a sum of three terms of spring forces; these terms represent the three springs to which the mass is attached. Spring forces are considered as positive when pointing towards the left (i.e. opposite to the direction of positive displacements), so they may be regarded as restoring forces. The net restoring force on each mass is shown in column 4.

Let us consider the first of the degenerate subspaces, $S^{(5)}$. The results of this are shown in Table 4 for its semi-subspace $S^{(5,1)}$ and Table 5 for its semi-subspace $S^{(5,2)}$. For subspace $S^{(5,1)}$, and by reference to Table 4, the restoring force *per unit value of b* is $(k_1 + k_2)$ at each location of mass.

Therefore, the stiffness $\widetilde{k}^{(5,1)}$ for subspace $S^{(5,1)}$ is given by

$$\widetilde{k}^{(5,1)} = (k_1 + k_2) \tag{49}$$

The mass upon which this force acts has the value m at each location of mass. Therefore, the characteristic equation for subspace $S^{(5,1)}$, given by $\widetilde{k}^{(5,1)} - (\omega^2)^{(5,1)} m = 0$, becomes

$$(k_1 + k_2) - (\omega^2)^{(5,1)} m = 0 \tag{50}$$

where ω^2 (the square of the natural circular frequency of the system) is the eigenvalue of the subspace. The solution for the eigenvalue of subspace $S^{(5,1)}$ immediately follows from Eq. (50):

$$(\omega^2)^{(5,1)} = \frac{k_1 + k_2}{m} \tag{51}$$

For subspace $S^{(5,2)}$, and by reference to Table 5, the restoring force per unit value of b is also $(k_1 + k_2)$ at each *active* location of mass. Therefore, the stiffness $\widetilde{k}^{(5,2)}$ for subspace $S^{(5,2)}$ is given by

$$\widetilde{k}^{(5,2)} = (k_1 + k_2) \tag{52}$$

which leads to the characteristic equation

$$(k_1 + k_2) - (\omega^2)^{(5,2)} m = 0 \tag{53}$$

with the solution

$$(\omega^2)^{(5,2)} = \frac{k_1 + k_2}{m} \tag{54}$$

The result for subspace $S^{(5,1)}$ is identical to that for subspace $S^{(5,2)}$ – compare Eqs. (51) and (54). Thus, through the use of operators $P^{(5,1)}$ and $P^{(5,2)}$ as given by Eqs. (11), the 2-dimensional subspace $S^{(5)}$ of the spring-mass system has been decomposed into two 1-dimensional subspaces $S^{(5,1)}$ and $S^{(5,2)}$, allowing the doubly-repeating solutions of subspace $S^{(5)}$ to be computed separately and more easily

Table 4
Restoring forces due to the application of the b displacements of $\Phi^{(5,1)}$.

Mass	Displacement b (+ve towards right)	Spring forces on mass (+ve towards left)	Net restoring force on mass
$m_1 = m$	$b_1 = +1$	$k_1 - k_2 + 2k_2$	$k_1 + k_2$
$m_2 = m$	$b_2 = -1$	$-k_1 - 2k_2 + k_2$	$-(k_1 + k_2)$
$m_3 = m$	$b_3 = -2$	$-k_2 - 2k_1 - k_2$	$-2(k_1 + k_2)$
$m_4 = m$	$b_4 = -1$	$k_2 - k_1 - 2k_2$	$-(k_1 + k_2)$
$m_5 = m$	$b_5 = +1$	$k_1 + 2k_2 - k_2$	$k_1 + k_2$
$m_6 = m$	$b_6 = +2$	$2k_1 + k_2 + k_2$	$2(k_1 + k_2)$

Table 5
Restoring forces due to the application of the b displacements of $\Phi^{(5,2)}$.

Mass	Displacement b (+ve towards right)	Spring forces on mass (+ve towards left)	Net restoring force on mass
$m_1 = m$	$b_1 = +1$	$k_1 + k_2 + 0k_2$	$k_1 + k_2$
$m_2 = m$	$b_2 = +1$	$0k_2 + k_1 + k_2$	$k_1 + k_2$
$m_3 = m$	$b_3 = 0$	$-k_2 + 0k_1 + k_2$	0
$m_4 = m$	$b_4 = -1$	$-k_2 + 0k_2 - k_1$	$-(k_1 + k_2)$
$m_5 = m$	$b_5 = -1$	$0k_2 - k_2 - k_1$	$-(k_1 + k_2)$
$m_6 = m$	$b_6 = 0$	$k_2 + 0k_1 - k_2$	0

within the smaller subspaces $S^{(5,1)}$ and $S^{(5,2)}$.

Let us next consider subspace $S^{(6)}$, with results shown in Table 6 for its semi-subspace $S^{(6,1)}$ and Table 7 for its semi-subspace $S^{(6,2)}$. For subspace $S^{(6,1)}$, and by reference to Table 6, the restoring force per unit value of b is $(k_1 + 3k_2)$ at each location of mass. Therefore, the stiffness $\tilde{k}^{(6,1)}$ for subspace $S^{(6,1)}$ is given by

$$\tilde{k}^{(6,1)} = (k_1 + 3k_2) \tag{55}$$

The mass upon which this force acts has the value m at each location of mass. Therefore, the characteristic equation for subspace $S^{(6,1)}$, given by $\tilde{k}^{(6,1)} - (\omega^2)^{(6,1)}m = 0$, becomes

$$(k_1 + 3k_2) - (\omega^2)^{(6,1)}m = 0 \tag{56}$$

The solution for the eigenvalue of subspace $S^{(6,1)}$ follows as:-

$$(\omega^2)^{(6,1)} = \frac{k_1 + 3k_2}{m} \tag{57}$$

For subspace $S^{(6,2)}$, and by reference to Table 7, the restoring force per unit value of b is also $(k_1 + 3k_2)$ at each *active* location of mass. Therefore, the stiffness $\tilde{k}^{(6,2)}$ for subspace $S^{(6,2)}$ is given by

$$\tilde{k}^{(6,2)} = (k_1 + 3k_2) \tag{58}$$

leading to the characteristic equation

$$(k_1 + 3k_2) - (\omega^2)^{(6,2)}m = 0 \tag{59}$$

with the solution

$$(\omega^2)^{(6,2)} = \frac{k_1 + 3k_2}{m} \tag{60}$$

Clearly, the result for subspace $S^{(6,1)}$ is identical to that for subspace $S^{(6,2)}$ – compare Eqs. (57) and (60). Thus, use of the new operators $P^{(6,1)}$ and $P^{(6,2)}$ (as given by Eqs. (20)) has enabled the 2-dimensional subspace $S^{(6)}$ of the spring-mass dynamic system to be treated as two independent 1-dimensional subspaces $S^{(6,1)}$ and $S^{(6,2)}$. In this way, the two equal eigenvalues of subspace $S^{(6)}$ have been obtained through the solution of two simple 1st-degree equations (actually, only one needed to be considered), instead of solution of a 2nd-degree characteristic equation (which would have been the case if subspace $S^{(6)}$ had not been decomposed further).

This 6 d.o.f. spring-mass system was analysed in earlier work [27], but without the benefit of operators $\{P^{(5,1)};\ P^{(5,2)}\}$ and $\{P^{(6,1)};\ P^{(6,2)}\}$. In that study, subspaces $S^{(1)}$ and $S^{(4)}$ were found to be each 1-dimensional, subspaces $S^{(2)}$ and $S^{(3)}$ were found to be null subspaces, and subspaces $S^{(5)}$ and $S^{(6)}$ were found to be each 2-dimensional and associated with doubly-repeating eigenvalues. Ref. [27] results for the six eigenvalues of the problem, and the subspaces from which they were calculated, are shown in Table 8. In that study, the eigenvalues of subspaces $S^{(5)}$ and $S^{(6)}$ were obtained by solving essentially 2nd-degree polynomials for the doubly repeating roots, whereas in the present study, they have been obtained by solving very simple 1st-degree equations of the

Table 6
Restoring forces due to the application of the b displacements of $\Phi^{(6,1)}$.

Mass	Displacement b (+ve towards right)	Spring forces on mass (+ve towards left)	Net restoring force on mass
$m_1 = m$	$b_1 = +1$	$k_1 + 0k_2 + 3k_2$	$k_1 + 3k_2$
$m_2 = m$	$b_2 = +1$	$0k_2 + k_1 + 3k_2$	$k_1 + 3k_2$
$m_3 = m$	$b_3 = -2$	$-3k_2 - 2k_1 - 3k_2$	$-2(k_1 + 3k_2)$
$m_4 = m$	$b_4 = +1$	$3k_2 + 0k_2 + k_1$	$k_1 + 3k_2$
$m_5 = m$	$b_5 = +1$	$0k_2 + k_1 + 3k_2$	$k_1 + 3k_2$
$m_6 = m$	$b_6 = -2$	$-3k_2 - 3k_2 - 2k_1$	$-2(k_1 + 3k_2)$

Table 7
Restoring forces due to the application of the b displacements of $\Phi^{(6,2)}$.

Mass	Displacement b (+ve towards right)	Spring forces on mass (+ve towards left)	Net restoring force on mass
$m_1 = m$	$b_1 = +1$	$k_1 + 2k_2 + k_2$	$k_1 + 3k_2$
$m_2 = m$	$b_2 = -1$	$-2k_2 - k_1 - k_2$	$-(k_1 + 3k_2)$
$m_3 = m$	$b_3 = 0$	$k_2 + 0k_1 - k_2$	0
$m_4 = m$	$b_4 = +1$	$k_2 + 2k_2 + k_1$	$k_1 + 3k_2$
$m_5 = m$	$b_5 = -1$	$-2k_2 - k_2 - k_1$	$-(k_1 + 3k_2)$
$m_6 = m$	$b_6 = 0$	$k_2 + 0k_1 - k_2$	0

semi-subspaces $\{S^{(5,1)};\ S^{(5,2)}\}$ and $\{S^{(6,1)};\ S^{(6,2)}\}$. The results of the present study are shown in column 3 of Table 8. In the present study, we are not concerned with the 1-dimensional subspaces $S^{(1)}$ and $S^{(4)}$ that are known to feature distinct solutions (as opposed to the doubly repeating solutions of degenerate subspaces $S^{(5)}$ and $S^{(6)}$), so in column 3 pertaining to the present study, no values for subspaces $S^{(1)}$ and $S^{(4)}$ are shown.

Focusing on subspaces $S^{(5)}$ and $S^{(6)}$, it is evident from Table 8 that the agreement between the present study and existing results in the literature is 100 %, showing that the newly proposed operators for the decomposition of subspace $S^{(5)}$ (as given by Eqs. (11)) and the decomposition of subspace $S^{(6)}$ (as given by Eqs. (20)) can be relied upon to simplify the computation of the doubly repeating solutions that are always associated with subspaces $S^{(5)}$ and $S^{(6)}$ of problems with C_{6v} symmetry. An added benefit of these operators is their ability to automatically separate the modes of a given degenerate subspace (which occur in pairs of equal frequency) into two distinct sets with orthogonal symmetry properties, such that modes of the same set have the same symmetry properties (as was illustrated in Figs. 4 and 5). This separation of symmetries was not possible in the previous work, and will be illustrated further in the section that follows.

5. Symmetries of the vibration modes of a 3-way hexagonal plane grid

Fig. 8 shows a regular hexagonal grid (in the horizontal plane) simply supported at the vertices of the hexagon, and comprising three systems of equi-spaced members, each system having 7 members that are parallel to two opposite sides of the hexagon. Grid members intersect rigidly at 37 joints (including the six supported joints), the whole arrangement of members and supports having the full C_{6v} symmetry of a regular hexagon.

Concentrated masses are located at the 24 joints that form the vertices of seven hexagonal cells; these 24 joints are numbered as shown. Clearly, the pattern of masses conforms to the C_{6v} symmetry of the structural configuration. We would like to determine the character of the modes (specifically symmetry properties) as the grid undergoes small transverse free vibrations, assuming the dominant freedoms of the grid are the vertical displacements $\{v_1,\ v_2,\ \dots,\ v_{24}\}$ of masses $\{m_1,\ m_2,\ \dots,\ m_{24}\}$ at joints $\{1,\ 2,\ \dots,\ 24\}$.

Following the group-theoretic procedure illustrated in the previous example, we apply each of the idempotents $P^{(1)}$, $P^{(2)}$, $P^{(3)}$ and $P^{(4)}$ of symmetry group C_{6v} – see Eqs. (2) to (5) – on each of the 24 freedoms $\{v_1,\ v_2,\ \dots,\ v_{24}\}$ of the grid, and select a set of independent symmetry-adapted freedoms as the basis vectors of the associated subspace. It is found that subspaces $S^{(1)}$, $S^{(2)}$, $S^{(3)}$ and $S^{(4)}$ have $\{3,\ 1,\ 1,\ 3\}$ basis vectors respectively, which are therefore the dimensions of these subspaces. The results for these four subspaces are as follows:

Subspace $S^{(1)}$

$$\Phi_1^{(1)} = v_1 + v_3 + v_{11} + v_5 + v_9 + v_7 + v_2 + v_6 + v_{10} + v_{12} + v_4 + v_8 \tag{61a}$$

$$\Phi_2^{(1)} = v_{13} + v_{14} + v_{15} + v_{16} + v_{17} + v_{18} \tag{61b}$$

Table 8
Results of the present study versus those from the literature [27].

Subspace $S^{(i)}$	Eigenvalue $(\omega^2)^{(i)}$ [Ref. 24]	Eigenvalue $(\omega^2)^{(i)}$ [Present study]
$S^{(1)}$	$(\omega^2)^{(1)} = \frac{k_1}{m}$	—
$S^{(4)}$	$(\omega^2)^{(4)} = \frac{k_1 + 4k_2}{m}$	—
$S^{(5)}$	$(\omega^2)_1^{(5)} = \frac{k_1 + k_2}{m}$	$S^{(5,1)}$: $(\omega^2)^{(5,1)} = \frac{k_1 + k_2}{m}$
	$(\omega^2)_2^{(5)} = \frac{k_1 + k_2}{m}$	$S^{(5,2)}$: $(\omega^2)^{(5,2)} = \frac{k_1 + k_2}{m}$
$S^{(6)}$	$(\omega^2)_1^{(6)} = \frac{k_1 + 3k_2}{m}$	$S^{(6,1)}$: $(\omega^2)^{(6,1)} = \frac{k_1 + 3k_2}{m}$
	$(\omega^2)_2^{(6)} = \frac{k_1 + 3k_2}{m}$	$S^{(6,2)}$: $(\omega^2)^{(6,2)} = \frac{k_1 + 3k_2}{m}$

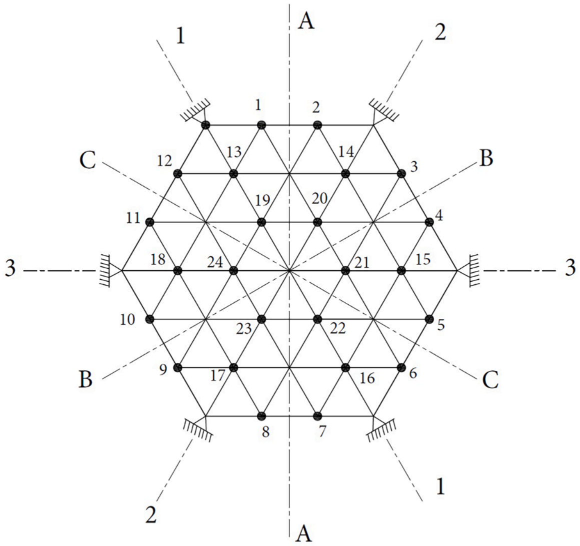


Fig. 8. A 24 d.o.f. hexagonal plane grid.

$$\Phi_3^{(1)} = v_{19} + v_{20} + v_{21} + v_{22} + v_{23} + v_{24} \tag{61c}$$

Subspace $S^{(2)}$

$$\Phi_1^{(2)} = v_1 + v_3 + v_{11} + v_5 + v_9 + v_7 - v_2 - v_6 - v_{10} - v_{12} - v_4 - v_8 \tag{62}$$

Subspace $S^{(3)}$

$$\Phi_1^{(1)} = v_1 - v_3 - v_{11} + v_5 + v_9 - v_7 + v_2 + v_6 + v_{10} - v_{12} - v_4 - v_8 \tag{63}$$

Subspace $S^{(4)}$

$$\Phi_1^{(4)} = v_1 - v_3 - v_{11} + v_5 + v_9 - v_7 - v_2 - v_6 - v_{10} + v_{12} + v_4 + v_8 \tag{64a}$$

$$\Phi_2^{(4)} = v_{13} - v_{14} - v_{18} + v_{15} + v_{17} - v_{16} \tag{64b}$$

$$\Phi_3^{(4)} = v_{19} - v_{20} - v_{24} + v_{21} + v_{23} - v_{22} \tag{64c}$$

The above results mean that, out of the 24 vibration modes of the system, subspaces $S^{(1)}$, $S^{(2)}$, $S^{(3)}$ and $S^{(4)}$ will contribute $\{3,\ 1,\ 1,\ 3\}$ vibration modes respectively, a total of only 8. The rest (i.e. 16 vibration modes) will be contributed by the degenerate subspaces $S^{(5)}$ and $S^{(6)}$.

Applying idempotent $P^{(5)}$ (Eq. (6)) to the 24 freedoms $\{v_1,\ v_2,\ \ldots,\ v_{24}\}$ results in 8 independent symmetry-adapted freedoms (hence 8 basis vectors) for subspace $S^{(5)}$. Similarly, applying idempotent $P^{(6)}$ (Eq. (7)) to the 24 freedoms results in 8 basis vectors for subspace $S^{(6)}$. Subspaces $S^{(5)}$ and $S^{(6)}$ are therefore each 8-dimensional. The associated eigenvalue problem is 8-dimensional in each case, leading to an 8 × 8 determinant whose vanishing condition results in an 8th-degree polynomial characteristic equation, solution of which yields eight eigenvalues of the subspace in question, occurring as four doubly repeating roots. Clearly, the computations for subspaces $S^{(5)}$ and $S^{(6)}$ require considerably more effort than those of subspaces $S^{(1)}$, $S^{(2)}$, $S^{(3)}$ and $S^{(4)}$. Moreover, the symmetries of the basis vectors of subspaces $S^{(5)}$ and $S^{(6)}$ do not necessarily reflect the actual symmetries of the vibration modes of subspaces $S^{(5)}$ and $S^{(6)}$.

On the other hand, use of the newly proposed operators $\{P^{(5,1)};\ P^{(5,2)}\}$ and $\{P^{(6,1)};\ P^{(6,2)}\}$ – refer to (Eqs. (11) and (20)) – automatically splits the 8-dimensional subspaces $S^{(5)}$ and $S^{(6)}$ into mutually orthogonal 4-dimensional subspaces $\{S^{(5,1)};\ S^{(5,2)}\}$ and $\{S^{(6,1)};\ S^{(6,2)}\}$, thus simplifying the computations. Subspaces $S^{(5,1)}$ and $S^{(5,2)}$ yield identical sets of 4 eigenvalues (i.e. the repeating solutions of the parent subspace $S^{(5)}$). Similarly, subspaces $S^{(6,1)}$ and $S^{(6,2)}$ yield identical sets of 4 eigenvalues (i.e. the repeating solutions of the parent subspace $S^{(6)}$).

Thus, if the objective is to obtain eigenvalues (i.e. natural frequencies of vibration) of the grid, we only need to consider subspace $S^{(5,1)}$ (which is 4-dimensional) to generate all 8 natural frequencies of subspace $S^{(5)}$, and subspace $S^{(6,1)}$ (also 4-dimensional) to generate all 8 natural frequencies of subspace $S^{(6)}$. Clearly, use of the operators $\{P^{(5,1)};\ P^{(5,2)}\}$ and $\{P^{(6,1)};\ P^{(6,2)}\}$ affords a very significant reduction in the computational effort involved in calculating the 16 eigenvalues (natural frequencies) associated with the degenerate subspaces $S^{(5)}$ and $S^{(6)}$.

As will be illustrated in due course, the symmetries of the basis vectors of the split subspaces $\left\{S^{(5,1)};\ S^{(5,2)}\right\}$ and $\left\{S^{(6,1)};\ S^{(6,2)}\right\}$ are a reflection of the actual symmetries of the vibration modes of subspaces $S^{(5)}$ and $S^{(6)}$, giving a more complete understanding of the vibration characteristics of the hexagonal grid.

Let us now apply each of the operators $P^{(5,1)}$, $P^{(5,2)}$, $P^{(6,1)}$ and $P^{(6,2)}$, as given by Eqs. (11) and (20), on each of the 24 freedoms $\{v_1, v_2, \ldots, v_{24}\}$ of the grid. This gives 24 symmetry-adapted freedoms for each of subspaces $S^{(5,1)}$, $S^{(5,2)}$, $S^{(6,1)}$ and $S^{(6,2)}$, only four of which

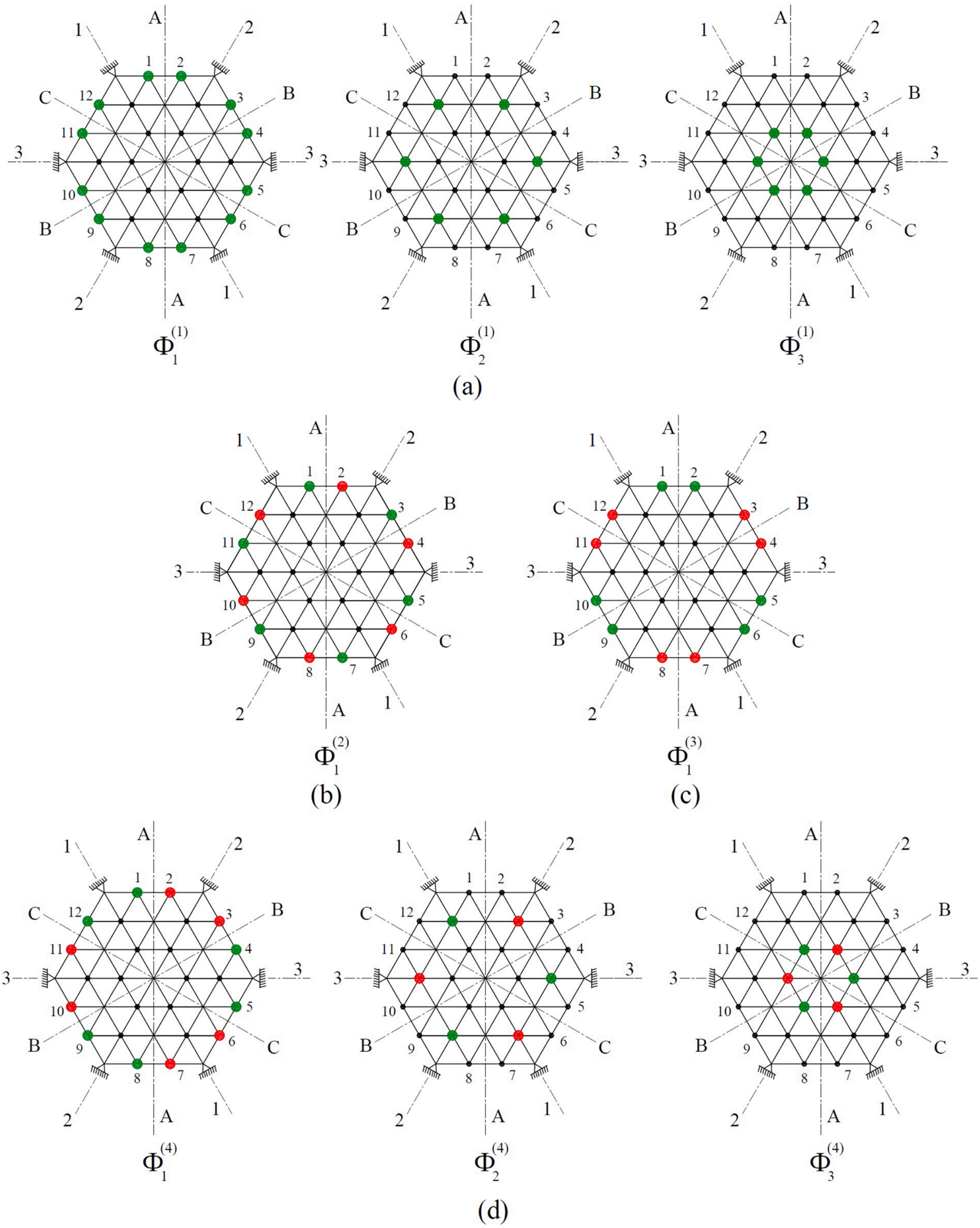


Fig. 9. Plots of basis vectors of the hexagonal plane grid: (a) subspace $S^{(1)}$; (b) subspace $S^{(2)}$; (c) subspace $S^{(3)}$; (d) subspace $S^{(4)}$. (For an interpretation of the references to colour in this figure, the reader is referred to the web version of this article.)

may be shown to be independent. From the results, we may choose subspace basis vectors as follows:

Subspace $S^{(5,1)}$

$$\Phi_1^{(5,1)} = 2v_1 + v_3 + v_{11} - v_5 - v_9 - 2v_7 - 2v_2 + v_6 + v_{10} - v_{12} - v_4 + 2v_8 \quad (65a)$$

$$\Phi_2^{(5,1)} = 2v_3 + v_5 + v_1 - v_7 - v_{11} - 2v_9 - 2v_{12} + v_4 + v_8 - v_{10} - v_2 + 2v_6 \quad (65b)$$

$$\Phi_3^{(5,1)} = v_{13} - v_{14} - 2v_{15} - v_{16} + v_{17} + 2v_{18} \quad (65c)$$

$$\Phi_4^{(5,1)} = v_{19} - v_{20} - 2v_{21} - v_{22} + v_{23} + 2v_{24} \quad (65d)$$

Subspace $S^{(5,2)}$

$$\Phi_1^{(5,2)} = 2v_1 + v_3 + v_{11} - v_5 - v_9 - 2v_7 + 2v_2 - v_6 - v_{10} + v_{12} + v_4 - 2v_8 \quad (66a)$$

$$\Phi_2^{(5,2)} = 2v_3 + v_5 + v_1 - v_7 - v_{11} - 2v_9 + 2v_{12} - v_4 - v_8 + v_{10} + v_2 - 2v_6 \quad (66b)$$

$$\Phi_3^{(5,2)} = v_{13} + v_{14} - v_{16} - v_{17} \quad (66c)$$

$$\Phi_4^{(5,2)} = v_{19} + v_{20} - v_{22} - v_{23} \quad (66d)$$

Subspace $S^{(6,1)}$

$$\Phi_1^{(6,1)} = 2v_1 - v_3 - v_{11} - v_5 - v_9 + 2v_7 + 2v_2 - v_6 - v_{10} - v_{12} - v_4 + 2v_8 \quad (67a)$$

$$\Phi_2^{(6,1)} = 2v_3 - v_5 - v_1 - v_7 - v_{11} + 2v_9 + 2v_{12} - v_4 - v_8 - v_{10} - v_2 + 2v_6 \quad (67b)$$

$$\Phi_3^{(6,1)} = v_{13} + v_{14} - 2v_{15} + v_{16} + v_{17} - 2v_{18} \quad (67c)$$

$$\Phi_4^{(6,1)} = v_{19} + v_{20} - 2v_{21} + v_{22} + v_{23} - 2v_{24} \quad (67d)$$

Subspace $S^{(6,2)}$

$$\Phi_1^{(6,2)} = 2v_1 - v_3 - v_{11} - v_5 - v_9 + 2v_7 - 2v_2 + v_6 + v_{10} + v_{12} + v_4 - 2v_8 \quad (68a)$$

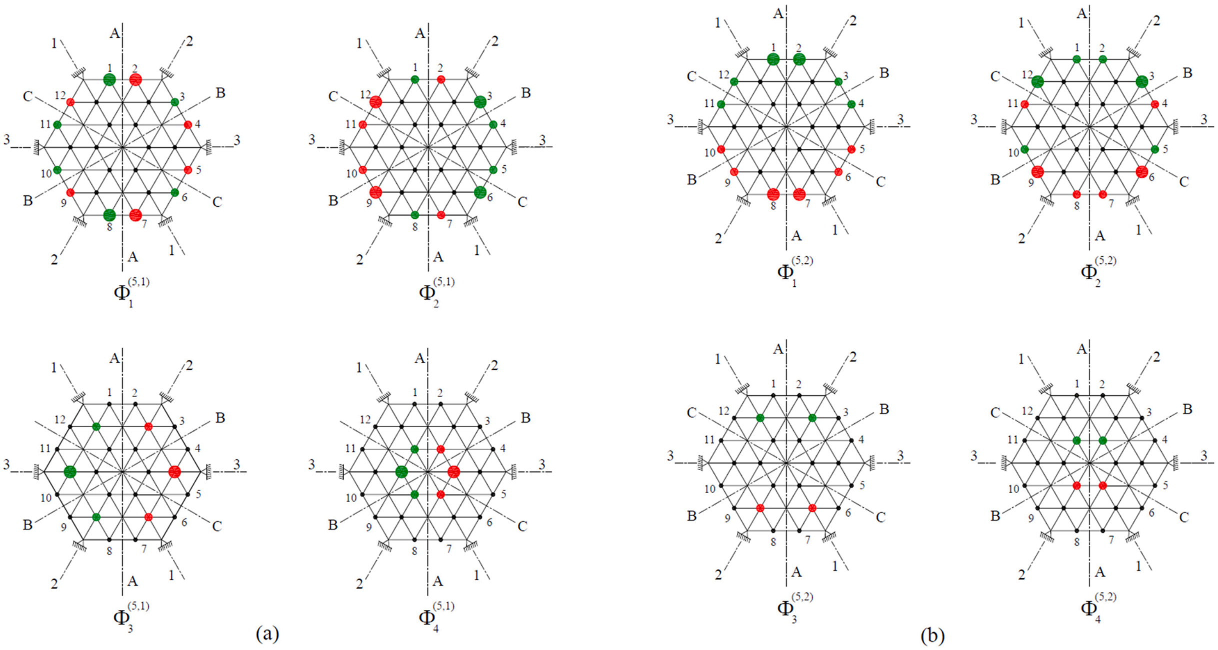


Fig. 10. Plots of basis vectors of the hexagonal plane grid: (a) subspace $S^{(5,1)}$; (b) subspace $S^{(5,2)}$. (For an interpretation of the references to colour in this figure, the reader is referred to the web version of this article.)

$$\Phi_2^{(6,2)} = 2v_3 - v_5 - v_1 - v_7 - v_{11} + 2v_9 - 2v_{12} + v_4 + v_8 + v_{10} + v_2 - 2v_6 \tag{68b}$$

$$\Phi_3^{(6,2)} = v_{13} - v_{14} + v_{16} - v_{17} \tag{68c}$$

$$\Phi_4^{(6,2)} = v_{19} - v_{20} + v_{22} - v_{23} \tag{68d}$$

As expected, the dimension of all four subspaces – $S^{(5,1)}$, $S^{(5,2)}$, $S^{(6,1)}$ and $S^{(6,2)}$ – is 4. This immediately implies that each of these subspaces will have 4 modes of vibration.

The symmetry types of all 24 modes of vibration may be determined by plotting the basis vectors of all eight subspaces, as given by Eqs. (61–68). Figs. 9–11 depict these plots. In the plots, green dots denote a positive value of v (i.e. a downward displacement), while red dots denote a negative value of v (i.e. an upward displacement); the smaller dots denote a coefficient of ± 1, while the larger dots denote a coefficient of ± 2. For subspaces $S^{(5,1)}$, $S^{(5,2)}$, $S^{(6,1)}$ and $S^{(6,2)}$, the symmetries of the plots in Figs. 10 and 11 exactly match those that were established for the spring-mass system – refer to Figs. 4 and 5. This shows that, regardless of the type of structural system under consideration, as long as the system has C_{6v} symmetry, the operators $P^{(5,1)}$, $P^{(5,2)}$, $P^{(6,1)}$ and $P^{(6,2)}$ will yield consistent predictions of the two symmetries that are mixed within the degenerate subspaces $S^{(5)}$ and $S^{(6)}$, at the same time separating out these symmetries.

Table 9 summarizes the symmetries of all subspaces. This information provides a valuable overview of the vibration behaviour of the grid, before any detailed computations for natural frequencies are carried out. It means that, in advance of such computations, we already know all the symmetries that will occur, and the number of vibration modes that will have each symmetry type. We also know in advance the vibration modes that have the same sets of frequencies but different mode shapes, which are the modes of subspaces $S^{(5,1)}$ and $S^{(5,2)}$, and the modes of subspaces $S^{(6,1)}$ and $S^{(6,2)}$.

6. Computation of natural frequencies and mode shapes

The procedure for applying the subspace basis vectors to the actual computation of natural frequencies was illustrated in Section 4 with reference to the vibration of a spring-mass system, based on the stiffness formulation. These steps have also been illustrated in detail in the various group-theoretic studies conducted to date [1,3,14,15,28]. Here, and for the sake of completeness, we will summarize the procedure with reference to the 24 d.o.f. hexagonal grid of the previous section, based on the flexibility formulation of the equilibrium equations.

Let us consider a subspace $S^{(\mu)}$ spanned by r basis vectors $\left\{\Phi_1^{(\mu)},\ \Phi_2^{(\mu)},\ \ldots,\ \Phi_r^{(\mu)}\right\}$. Similar to the concept of a symmetry-adapted stiffness coefficient introduced in Section 4, a symmetry-adapted flexibility coefficient $\tilde{a}_{ij}$ ($i = 1,\ 2,\ \ldots,\ r; j = 1,\ 2,\ \ldots,\ r$) is defined as the vertical displacement per unit value of the force coefficient (analogous to the displacement coefficient b of Section 4) at the first of the joints of basis vector Φ_i, caused by unit vertical forces applied simultaneously at *all* the joints of basis vector Φ_j. The assembly of all $\tilde{a}_{ij}$ coefficients constitutes the symmetry-adapted flexibility matrix $\tilde{A}^{(\mu)}$ for subspace $S^{(\mu)}$. The application of unit vertical forces at the joints of basis vector Φ_j must be in accordance with the coefficients of the basis vector Φ_j.

For example, considering subspace $S^{(4)}$ of the hexagonal grid of the last section (this has three basis vectors, hence $r = 3$), the symmetry-adapted flexibility coefficient $\tilde{a}_{2,3}$ is the vertical displacement at joint 13 (noting that the first coordinate of $\Phi_2^{(4)}$ is v_{13} – see Eq. (64b)) due to the simultaneous application of unit forces $\{+1,\ -1,\ -1,\ +1,\ +1,\ -1\}$ at joints $\{19,\ 20,\ 24,\ 21,\ 23,\ 22\}$ respectively (noting that the coordinates of $\Phi_3^{(4)}$ are $\{+v_{19}, -v_{20}, -v_{24}, +v_{21}, +v_{23}, -v_{22}\}$ – see Eq. (64c)). In this way, all nine coefficients making up the flexibility matrix $\tilde{A}^{(4)}$ can easily be assembled.

While the conventional flexibility matrix of the system $[A]$ is always a symmetric matrix ($a_{ij} = a_{ji}$), it must be noted that symmetry-adapted flexibility matrices $\tilde{A}^{(\mu)}$ are generally not symmetric ($\tilde{a}_{ij} \neq \tilde{a}_{ji}$). The same applies to symmetry-adapted stiffness matrices ($\tilde{k}_{ij} \neq \tilde{k}_{ji}$).

The symmetry-adapted diagonal mass matrix $\tilde{M}^{(\mu)}$ for subspace $S^{(\mu)}$ comprises diagonal elements $\tilde{m}_{ii}$ ($i = 1,\ 2,\ \ldots,\ r$), which are the values of the mass at each of the joints of basis vector Φ_i. Considering the 24 d.o.f. hexagonal grid in Fig. 8 as an illustrative example, let us assume equal masses m_1 are assigned at each location of the joint set $\{1,\ 2,\ \ldots,\ 12\}$, equal masses m_2 are assigned at each location of the joint set $\{13,\ 14,\ \ldots,\ 18\}$, and equal masses m_3 are assigned at each location of the joint set $\{19,\ 20,\ \ldots,\ 24\}$. This pattern of masses clearly conforms to the C_{6v} symmetry of the structural configuration. The symmetry-adapted mass matrix for subspace $S^{(4)}$, for example, would be given by $\tilde{M}^{(4)} = \text{diag}\ [m_1,\ m_2,\ m_3]$, since the three basis vectors $\Phi_1^{(4)}$, $\Phi_2^{(4)}$ and $\Phi_3^{(4)}$ of subspace $S^{(4)}$ are associated with joints having masses m_1, m_2 and m_3 respectively.

For each subspace $S^{(\mu)}$ of a free vibration problem belonging to a given symmetry group G, and ignoring the effects of damping, the eigenvalues λ are obtained from the vanishing condition

$$\left|\tilde{A}^{(\mu)} - \lambda\left[\tilde{M}^{(\mu)}\right]^{-1}\right| = 0 \tag{69}$$

where $\tilde{A}^{(\mu)}$ and $\tilde{M}^{(\mu)}$ are the associated symmetry-adapted flexibility and mass matrices. Expanding the determinant in Eq. (69) leads to an rth-degree characteristic polynomial, solution of which yields r eigenvalues of subspace $S^{(\mu)}$. It turns out that eigenvalues of the

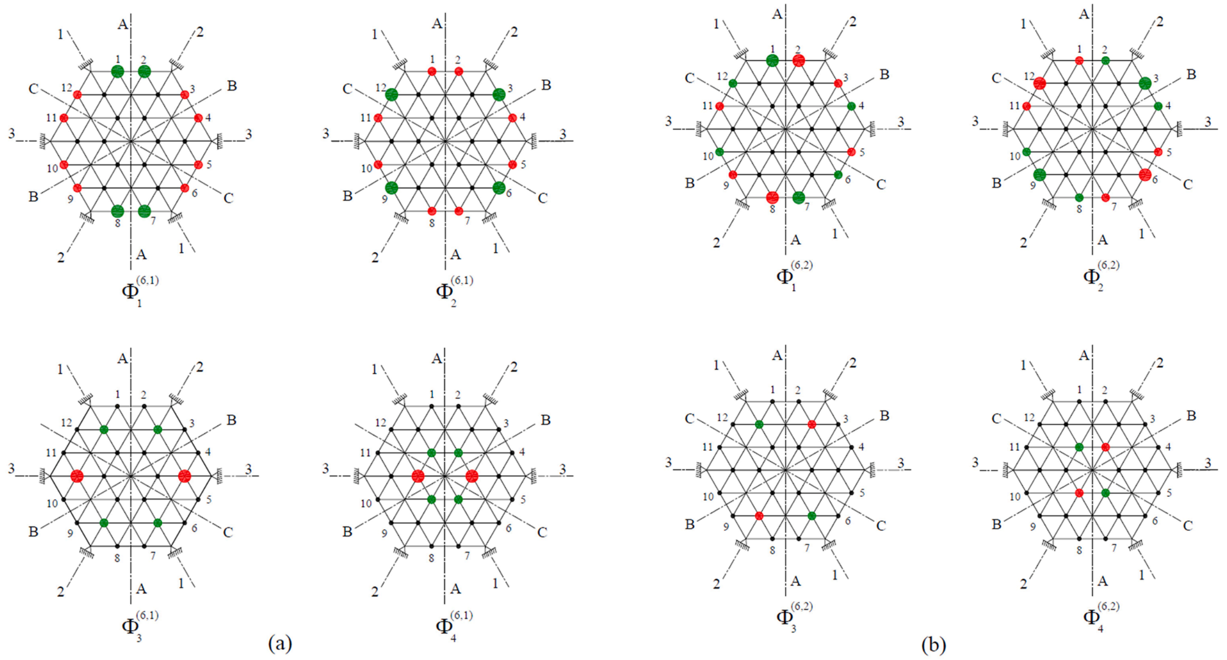


Fig. 11. Plots of basis vectors of the hexagonal plane grid: (a) subspace $S^{(6,1)}$; (b) subspace $S^{(6,2)}$. (For an interpretation of the references to colour in this figure, the reader is referred to the web version of this article.)

Table 9
Symmetries of the modes of the subspaces of the hexagonal grid.

Subspace	Figure No.	Symmetries of modes
$S^{(1)}$	9(a)	symmetric about all axes (full C_{6v} symmetry)
$S^{(2)}$	9(b)	antisymmetric about all axes
$S^{(3)}$	9(c)	symmetric about *A-A, B-B, C-C*; antisymmetric about *1-1,2-2,3-3*
$S^{(4)}$	9(d)	antisymmetric about *A-A, B-B, C-C*; symmetric about *1-1,2-2,3-3*
$S^{(5,1)}$	10(a)	antisymmetric about *A-A*; symmetric about *3-3* (*A-A* and *3-3* are ⊥)
$S^{(5,2)}$	10(b)	symmetric about *A-A*; antisymmetric about *3-3* (*A-A* and *3-3* are ⊥)
$S^{(6,1)}$	11(a)	symmetric about *A-A*; symmetric about *3-3* (*A-A* and *3-3* are ⊥)
$S^{(6,2)}$	11(b)	antisymmetric about *A-A*; antisymmetric about *3-3* (*A-A* and *3-3* are ⊥)

subspaces are also eigenvalues of the system as a whole, so the natural frequencies of the system immediately follow. Thus, by solving a series of independent smaller eigenvalue problems (as afforded by group-theoretic decomposition using subspace idempotents), all the natural frequencies of the original problem can be obtained, which represents a considerable simplification of the analysis.

Substituting an eigenvalue λ into the subspace eigenvalue equation:-

$$\left[\widetilde{A}^{(\mu)} - \lambda\left[\widetilde{M}^{(\mu)}\right]^{-1}\right]\{\Psi\} = \{0\} \tag{70}$$

allows the corresponding eigenvector $\{\Psi\}$ to be obtained. The eigenvector obtained from Eq. (70) is r-dimensional (i.e. it has the dimension of the subspace in question). By allocating each of the r components of $\{\Psi\}$ to all the coordinates associated with the corresponding basis vector, we transform the r-dimensional vector $\{\Psi\}$ in the space of $S^{(\mu)}$ into an n-dimensional vector $\{U\}$ in the full space of the problem, thus obtaining the actual mode shape.

For example, if one of the eigenvectors of the 3-dimensional subspace $S^{(4)}$ of the 24 d.o.f. grid (Fig. 8) is obtained as $\{\psi_1, \ \psi_2, \ \psi_3\}^T$, where ψ_1, ψ_2 and ψ_3 are numerical values, then the eigenvector $\{U\}$ in the 24-dimensional space of the grid (which is the mode shape of the grid corresponding to the eigenvalue λ) is obtained by simply assigning the value ψ_1 to each of the 12 joint locations $J_1 = \{1, \ 2, \ \ldots, \ 12\}$, the value ψ_2 to each of the 6 joint locations $J_2 = \{13, \ 14, \ \ldots, \ 18\}$, and the value ψ_3 to each of the 6 joint locations $J_3 = \{19, \ 20, \ \ldots, \ 24\}$, since J_1, J_2 and J_3 are the joint sets associated with the basis vectors $\Phi_1^{(4)}$, $\Phi_2^{(4)}$ and $\Phi_3^{(4)}$ of subspace $S^{(4)}$. For a full illustration of the procedure, reference should be made to earlier work [28].

7. Concluding remarks

The C_{6v} symmetry group describes the symmetry of a 6-sided regular polygon, such configurations finding application in long-span roofing (cable nets, lattice shells, truss domes, plane grids and space grids) and communication infrastructure (radio telescopes, antennae and deployable satellites). In this paper, two pairs of explicit operators for the decomposition of the two degenerate subspaces of structural configurations belonging to the symmetry group C_{6v} have been proposed, and shown to have the properties of idempotents of a symmetry group. These pairs of operators, namely $\{P^{(5,1)};\ P^{(5,2)}\}$ for subspace $S^{(5)}$ and $\{P^{(6,1)};\ P^{(6,2)}\}$ for subspace $S^{(6)}$, cannot be derived from the usual expression for deducing the idempotents of a symmetry group from its character table, and so had to be sought subject to a prescribed set of conditions. Unlike existing formulae, these operators are simple algebraic expressions which do not involve the use of matrices of the irreducible representations of the symmetry group. The presentation of such simple operators is the main contribution of this paper.

The operators have been applied to the problem of the free vibration of a spring-mass system with 6 d.o.f, for which subspace basis vectors have been derived, and natural frequencies calculated. The natural frequencies obtained using the new operators have been shown to be exactly identical to those that were obtained in previous work without the use of these operators, thus validating the proposed operators. The operators have also been applied to the study of the small transverse vibrations of a 24 d.o.f hexagonal grid system, for which all possible symmetries of modes, and the subspaces to which they belong, have been determined.

By acting on the conventional freedoms of a structural system with C_{6v} symmetry, the pairs of operators $\{P^{(i,1)};\ P^{(i,2)}\}$ (where $i = 5$ for subspace $S^{(5)}$ and $i = 6$ for subspace $S^{(6)}$) have the ability to generate two sets of basis vectors that are orthogonal to each other, effectively decomposing the degenerate subspace $S^{(i)}$ (associated with doubly-repeating solutions of the problem) into two independent semi-subspaces $S^{(i,1)}$ and $S^{(i,2)}$, each having half the dimension of the parent subspace $S^{(i)}$. This separation of repeating solutions allows the computations to be performed more easily without the numerical difficulties usually associated with coincident solutions. In eigenvalue vibration problems, semi-subspaces $S^{(i,1)}$ and $S^{(i,2)}$ yield identical sets of natural frequencies (which occur as repeating frequencies in the parent subspace), so to obtain all the eigenvalues of the parent subspace $S^{(i)}$, we only need to consider one semi-subspace.

A second merit of the pairs of operators $\{P^{(i,1)};\ P^{(i,2)}\}$ is their ability to separate the modes in the parent subspace $S^{(i)}$ into two orthogonal sets of modes, where modes in one set all have the same symmetry, which is different to the symmetry of the other set. In the parent subspace $S^{(i)}$, these modes are mixed together. The separation of modes afforded by operators $\{P^{(5,1)};\ P^{(5,2)}\}$ and $\{P^{(6,1)};\ P^{(6,2)}\}$ reveals distinctive characteristics of the modes of the degenerate subspaces $S^{(5)}$ and $S^{(6)}$. For a system with hexagonal C_{6v} symmetry, with reference to two perpendicular planes of symmetry $m1$ (bisecting two opposite sides of the hexagon) and $m2$ (passing through two opposite corners of the hexagon), the following always holds:

(i) For subspace $S^{(5)}$, one set of modes will be symmetric about $m1$ and antisymmetric about $m2$, while the other set will be antisymmetric about $m1$ and symmetric about $m2$.
(ii) For subspace $S^{(6)}$, one set of modes will be symmetric about both $m1$ and $m2$, while the other set will be antisymmetric about both $m1$ and $m2$.

Thus, not only do the proposed operators simplify the computation of repeating frequencies of structural systems with C_{6v} symmetry, but they have also provided a better understanding of the character of the vibration modes associated with repeating frequencies. In the context of linear eigenvalue vibration analysis, the only essential requirement is that the structural configuration must belong to the C_{6v} symmetry group. For the decomposition of degenerate subspaces of higher-order symmetry groups, different sets of operators specific to those symmetry groups are required. These are currently being developed on the basis of a similar set of criteria, and following the same general strategy, as that adopted in the present paper.

CRediT authorship contribution statement

Alphose Zingoni: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Resources, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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