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Decomposition of the degenerate subspace of C_{3v}-symmetric structural configurations

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ABSTRACT

Physical systems exhibiting symmetry properties may be conveniently studied using the mathematics of group theory. In structural mechanics, group theory has been successfully employed to simplify problems of the bifurcation, stability, statics, kinematics and vibration of symmetric configurations of space frames, space truss domes, double-layer and triple-layer space grids, plates and cable-net systems. Besides significantly reducing computational effort, group theory affords deeper insights on structural behaviour, and a better understanding of complex structural phenomena. The key to group-theoretic simplification is the decomposition of the space of the problem into independent subspaces that are spanned by symmetry-adapted variables obtained by applying idempotents of the symmetry group on the normal variables of the problem. However, for degenerate subspaces of a symmetry group (i.e. subspaces associated with repeating solutions), the associated idempotents do not sufficiently decompose the problem. The aim of this paper is to present, for the C_{3v} symmetry group describing the symmetry of a regular 3-sided polygon, a pair of algebraic operators that fully decompose such subspaces. Compared with existing group-theoretic formulations, these operators offer an alternative approach that is simpler and more suited to practical engineering computations, and that affords clearer insights on the physical characteristics of the structural system (such as type of symmetries within the degenerate subspaces). The validity of the operators is confirmed through comparisons with results of eigenvalue vibration and stability problems drawn from the literature.

1. Introduction

Symmetry is very common in structural engineering and architecture. Besides its aesthetic appeal, symmetry can enhance the functionality of space. From a structural point of view, symmetry can be taken advantage of to simplify the analysis of the system, or to reduce the costs of assembly of the system. However, symmetry also attracts complications in structural behaviour, such as the occurrence of multiple critical points in bifurcation analysis (where two or more eigenvalues vanish simultaneously), and the coincidence or near-coincidence of eigenvalues in problems of the vibration or buckling of structures, both of which pose difficulties of numerical ill-conditioning of solution procedures in computational schemes [1,2]. Suitable tools are needed to facilitate the study of structural configurations with higher-order symmetries, and better understand the associated complex phenomena.

The set of symmetry elements describing the symmetry of a physical configuration constitutes a symmetry group. Group theory provides the mathematical tools for the study of such systems [3–5]. This allows the space of the problem to be decomposed into independent symmetry-adapted subspaces. Within the domain of structural mechanics, group theory has been successfully employed to simplify the study of the bifurcation of space trusses [1,6,7], the statics of space frames [8,9] and pin-jointed trusses [10], the vibration of cable nets [11, 12], layered space grids [13] and plates [14], the kinematics of skeletal structures [15–17], and the stability of frames [18–20] and origami [21]. Applications in computational structural mechanics were highlighted in a survey that was conducted fifteen years ago [22]. Interesting applications of group theory to the study of auxetic metamaterials have also been reported in more recent literature [23], while related symmetry considerations have been applied to the study of foldable conical origami structures with potential for use as energy absorption structures [24].

Besides significantly reducing computational effort, group theory affords deeper insights on structural behaviour, and a better understanding of complex structural phenomena (for instance, it explains why certain natural frequencies repeat in symmetric vibrating systems [11,

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13,25–27]). The key to group-theoretic simplification is the decomposition of the space of the symmetric problem into independent subspaces that are spanned by symmetry-adapted variables, allowing the problem to be broken down into smaller independent problems that are easier to study, or easier to analyse. By separating the computation of coincident eigenvalues into independent subspaces that "do not see each other", group theory also circumvents the numerical problems associated with computing closely-spaced or coincident solutions in the full space of the problem [1,2]. Group theory effectively *untangles* the symmetries.

According to the representation theory of symmetry groups [3,4], each independent symmetry-adapted subspace S is associated with an irreducible representation Γ of the symmetry group; if the symmetry group has k irreducible representations, then the number of independent symmetry-adapted subspaces will be k. In turn, each irreducible representation $\Gamma^{(i)}$ ($i = 1, 2, \ldots, k$) of the symmetry group is associated with a unique idempotent $P^{(i)}$ ($i = 1, 2, \ldots, k$), which is a very specific linear combination of the symmetry elements of the group, having the special property of nullifying all vectors that do not belong to the subspace $S^{(i)}$ of the irreducible representation $\Gamma^{(i)}$, and selecting only vectors that belong to the subspace $S^{(i)}$. Idempotents of any symmetry group satisfy the relation $P^{(i)}P^{(i)} = P^{(i)}$ for all i. More importantly, they have the property $P^{(i)}P^{(j)} = 0$ if $i \neq j$ (i.e. idempotents of different subspaces are *orthogonal* to each other).

Each subspace has its own characteristic symmetry properties which distinguish it from other subspaces [25]. As an example, the symmetry group C_{1v} describing the symmetry of configurations with one reflection plane (such as a simply supported beam with two equal point loads P equidistant from the centre of the beam) has two irreducible representations $\Gamma^{(1)}$ and $\Gamma^{(2)}$ with corresponding idempotents $P^{(1)} = 0.5(e+\sigma_v)$ and $P^{(2)} = 0.5(e-\sigma_v)$, the symmetry elements $\{e, \sigma_v\}$ denoting the identity operation and reflection operation respectively. The idempotent $P^{(1)}$ and $P^{(2)}$, by operating on the normal variables of the problem, split the space of the problem into a *symmetric* subspace $S^{(1)}$ and an *anti-symmetric* subspace $S^{(2)}$ respectively.

Taking the idempotent $P^{(i)}$ corresponding to the irreducible representation $\Gamma^{(i)}$ (and associated with the subspace $S^{(i)}$), and applying this to each of the n normal variables of the problem, we obtain n symmetry-adapted variables, of which say r_i are independent. The r_i independent symmetry-adapted variables may be taken as the basis vectors of subspace $S^{(i)}$. Thus, subspace $S^{(i)}$ is of dimension r_i, where $r_i \ll n$; the sum of the dimensions of all k subspaces is equal to n: that is, $r_1 + r_2 + \ldots + r_k = n$ [11,13,25,28].

For any 1-dimensional irreducible representation $\Gamma^{(i)}$ of a symmetry group (the dimension of $\Gamma^{(i)}$ is given by the first *character* of $\Gamma^{(i)}$ in the character table [3–5] of the group), the dimension r_i of the associated subspace $S^{(i)}$ is the smallest possible (i.e. no further decomposition of subspace $S^{(i)}$ is possible). However, for an m-dimensional irreducible representation (where m can be 2, 3, 4 or 5), the decomposition yielded by the application of idempotent $P^{(i)}$ results in a subspace $S^{(i)}$ that can still be decomposed further. Such *degenerate* subspaces are associated with repeating solutions (which, in the case of eigenvalue vibration problems, are repeating natural frequencies); the degree of repetition is equal to m. Irreducible representations of dimension 1 or 2 are typically associated with structural configurations belonging to cyclic (C) and dihedral (D) symmetry groups, whereas those of dimension greater than 2 are only encountered in the analysis of tetrahedral (T), octahedral (O) and icosahedral (I) configurations. It should be pointed out that in this contribution, we will only be concerned with structural configurations that have a fixed centre of symmetry (of which there are many such examples in structural engineering), so only point groups will be relevant. Translational symmetry will not be taken into account, and space groups are outside the scope of this study.

Fig. 1 shows double-layer grids (in plan and elevation) belonging to symmetry groups C_{3v} and C_{6v}, which characterise configurations with the symmetries of a regular (or equilateral) triangle (3 rotations and 3 reflections) and a regular hexagon (6 rotations and 6 reflections). The grid in Fig. 1(a) has 3 supported nodes and 13 unsupported nodes, while that in Fig. 1(b) has 6 supported nodes and 37 unsupported nodes; the unsupported nodes of each grid are numbered as shown.

By reference to standard texts on application of group theory to physical problems with symmetry (see, for example, Ref. [3]), one may note that the character table of symmetry group C_{3v} features three irreducible representations $\Gamma^{(1)}$, $\Gamma^{(2)}$ and $\Gamma^{(3)}$, the first two of which are 1-dimensional while $\Gamma^{(3)}$ is 2-dimensional. On the other hand, the character table of symmetry group C_{6v} features six irreducible representations: $\Gamma^{(1)}$ to $\Gamma^{(4)}$) are 1-dimensional, while $\Gamma^{(5)}$ and $\Gamma^{(6)}$ are 2-dimensional.

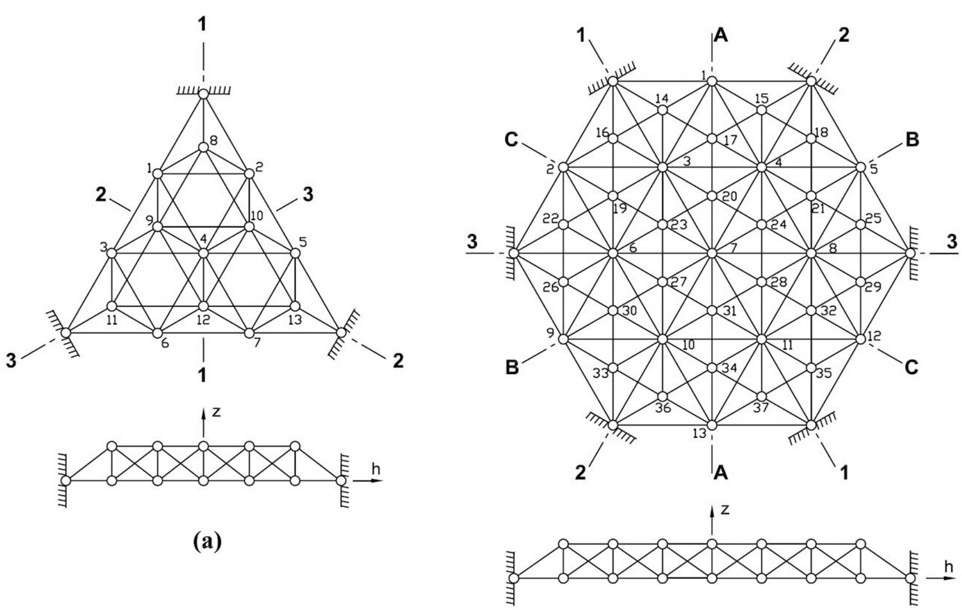

Fig. 1. Space grids with C_{nv} symmetry: (a) triangular grid (C_{3v} symmetry); (b) hexagonal grid (C_{6v} symmetry) [13].

From the above information, we deduce that problems involving the symmetry group C_{3v} will have two normal subspaces $S^{(1)}$ and $S^{(2)}$ corresponding to the 1-dimensional irreducible representations $\Gamma^{(1)}$ and $\Gamma^{(2)}$ of the symmetry group, and one degenerate subspace $S^{(3)}$ corresponding to the 2-dimensional irreducible representation $\Gamma^{(3)}$ of the symmetry group. Similarly, problems involving the symmetry group C_{6v} will have four normal subspaces $S^{(1)}$ to $S^{(4)}$ corresponding to the 1-dimensional irreducible representations $\Gamma^{(1)}$ to $\Gamma^{(4)}$ of the symmetry group, and two degenerate subspaces $S^{(5)}$ and $S^{(6)}$ corresponding to the 2-dimensional irreducible representations $\Gamma^{(5)}$ and $\Gamma^{(6)}$ of the symmetry group. In the present context, and as already pointed out, a degenerate subspace is one which is associated with solutions that repeat twice or more times, instead of being distinct from each other.

Clearly, if further decomposition of a degenerate subspace can be achieved, this would allow the doubly-repeating (or multiply-repeating) solutions of the subspace to be computed within separate smaller subspaces, not only circumventing the numerical difficulties of computing coincident solutions, but also reducing computational effort by reducing the size of the problem. Simple operators that further decompose the degenerate subspace $S^{(5)}$ of problems belonging to the symmetry group C_{4v} (which describes the symmetry of square configurations) were first introduced by the first author in a study of the vibration modes of symmetric layered space grids [13], and subsequently applied to the eigenvalue analysis of plates [14], plane grids [28] and plane frames [20], allowing the doubly-repeating eigenvalues of subspace $S^{(5)}$ to be obtained by consideration of only one of its semi-subspaces $S^{(5,1)}$ and $S^{(5,2)}$.

As far as the authors are aware of, operators that further decompose the degenerate subspaces of symmetry groups C_{3v} and C_{6v}, in terms of the symmetry elements of the group, have never been reported in the literature in a simple algebraic form that is easily applicable by structural engineers. General strategies for the splitting of a degenerate subspace $S^{(i)}$ may be seen in early texts on applications of group theory in physics and physical chemistry (see, for example, McWeeny [29]), and in other work that appeared later (Healey & Treacy [30]), but these required matrices of the irreducible representations to be written down first, and did not distinguish the symmetry types of the ensuing subspaces in a manner that has a clear engineering interpretation.

In this paper, we will present a simple pair of operators for the automatic decomposition of the degenerate subspace of problems belonging to the symmetry group C_{3v}. The difference with existing approaches (as presented in early literature) is that the operators that will be presented in this paper are simple algebraic linear combinations of the symmetry elements of the group; they do not involve matrices of irreducible representations, and are therefore very convenient to use in the course of practical engineering computations. While less general in the sense of being applicable only to a particular symmetry group, the proposed operators are readily applicable to the study and analysis of any structural problem that has C_{3v} symmetry, of which there are many in the form of cable-net systems, layered space grids, truss domes, lattice shells, plane frames and plane grids.

We will illustrate their application by reference to the small transverse vibrations of the double-layer space grid in Fig. 1(a). Vibration modes of this space grid were studied in earlier work [13], but without the benefit of the new operators. It should be noted that the usefulness of these operators extends beyond structural mechanics, as they can also be used to simplify problems in material science, physics and chemistry, as long as the problems have C_{3v} symmetry. The operators will be validated by consideration of the vibration analysis of a spring-mass system and the buckling analysis of a plane triangular frame. Results for eigenvalues and mode shapes obtained using the new operators will be compared with those previously obtained without the use of the operators.

It should be pointed out from the outset that the objective of this particular work is not so much to provide an improved computational procedure, but rather, to provide a mathematical formulation that allows structural engineers to better understand the physical behaviour of structural systems with C_{3v} symmetry, whether this behaviour is vibration, buckling or kinematics. Thus, the primary aim is to reveal the types of symmetries that occur in the degenerate subspace of systems belonging to the symmetry group C_{3v}, and to show how modes of equal frequency within the degenerate subspace can be separated from each other by use of simple algebraic operators, in that way simplifying the actual computation of repeating frequencies (if needed). We are not concerned with assessing the computational performance of the group-theoretic formulation in comparison with well-established numerical procedures such as the finite element method (FEM). Numerical methods such as FEM are suitable for numerical computations, but they do not afford the same type of insights and the deeper understanding of physical phenomena that the mathematics of group theory affords in studying problems with symmetry.

For instance, in analysing a symmetric space grid or a symmetric cable-net system for natural frequencies of vibration, FEM will give all sought values for natural frequencies, some of which are observed to occur in identical pairs, but FEM will not explain why this phenomenon of repeating frequencies occurs. On the other hand, and as will be illustrated in this paper, by decomposing the vector space of the problem into symmetry-adapted subspaces, *group theory allows us to predict the occurrence of such doubly-repeating frequencies before we even perform any detailed calculations*. These are the type of insights being referred to here. Such insights on physical phenomena have been described in detail in previous work of the first author [13,25,27]. That is why group theory has been used so fruitfully by physicists and physical chemists in studying phenomena in quantum mechanics, molecular vibrations, crystallography and other topics [31–33]. Similar benefits can be gained in the context of structural mechanics.

2. Idempotents of symmetry group C_{3v}

Symmetry operations are transformations which bring an object into coincidence with itself, and leaves it indistinguishable from its original configuration. In the double-layer grid shown in Fig. 1, nodes 1 to 7 are in the bottom layer, while nodes 8 to 13 are in the top layer, vertically above the centroids of the bottom triangles. The centre of symmetry is at node 4, through which the vertical axis of rotational symmetry of the configuration passes.

By reference to the upper diagram of Fig. 1(a), the symmetry operations of group C_{3v}, describing the symmetry of a regular 3-sided polygon, are $\{e, C_3, C_3^{-1}, \sigma_1, \sigma_2, \sigma_3\}$, where e is the identity element (equivalent to a rotation of 2π about the axis of rotational symmetry), C_3 and C_3^{-1} are clockwise and anticlockwise rotations of $2\pi/3$ about the axis of rotational symmetry, while σ_1, σ_2 and σ_3 are reflections in vertical planes $1-1$, $2-2$ and $3-3$ as shown.

Table 1 gives the group table for symmetry group C_{3v}. For a given symmetry group with elements $\{\alpha, \beta, \gamma, \ldots\}$, the group table comprises products $\alpha\beta$, generated by multiplying a symmetry element α from the left of the table by a symmetry element β from the top of the table (in that order). For example, $C_3^{-1}\sigma_2$ (i.e. an anticlockwise rotation of $2\pi/3$ followed by a reflection in the axis $2-2$) is equivalent to the symmetry

Table 1
Group table for symmetry group C_{3v}.

C_{3v}	e	C_3	C_3^{-1}	σ_1	σ_2	σ_3
e	e	C_3	C_3^{-1}	σ_1	σ_2	σ_3
C_3	C_3	C_3^{-1}	e	σ_2	σ_3	σ_1
C_3^{-1}	C_3^{-1}	e	C_3	σ_3	σ_1	σ_2
σ_1	σ_1	σ_3	σ_2	e	C_3^{-1}	C_3
σ_2	σ_2	σ_1	σ_3	C_3	e	C_3^{-1}
σ_3	σ_3	σ_2	σ_1	C_3^{-1}	C_3	e

operation σ_1 (a reflection in the axis $1-1$). The group table is very useful in writing down the product of any two symmetry elements of the group.

In general, the orthogonal idempotents of a symmetry group G can be written down directly from the character table of the group, using the relation [3–5]:

$$P^{(i)} = \frac{h_i}{h}\sum_{\sigma}\chi_i(\sigma^{-1})\sigma \quad (1)$$

where $P^{(i)}$ corresponds to $\Gamma^{(i)}$ (the ith irreducible representation of the symmetry group G), h_i is the dimension of $\Gamma^{(i)}$ (given by $h_i = \chi_i(e)$, the first character of the ith row of the character table), h is the order of the symmetry group (that is, the number of elements of G), χ_i is a character of $\Gamma^{(i)}$, σ is an element of G, and σ^{-1} its inverse. If $\Gamma^{(i)}$ is a 1-dimensional representation of G, then $h_i = 1$; if $\Gamma^{(i)}$ is a 2-dimensional representation of G, then $h_i = 2$, and so forth.

Each idempotent $P^{(i)}$, when applied to the variables of a problem, generates the symmetry-adapted variables of the corresponding subspace $S^{(i)}$ of the problem. Degenerate subspaces (i.e. those which have the potential for further decomposition) stem from irreducible representations of dimension 2 or greater (i.e. the $\Gamma^{(i)}$ for which $h_i \geq 2$). Eq. (1) only yields one idempotent $P^{(i)}$ for the degenerate subspace $S^{(i)}$, which can be used to generate the symmetry-adapted variables for subspace $S^{(i)}$. If further decomposition of a degenerate subspace is required (which is the objective of the present work), then additional operators for that purpose need to be sought.

The three idempotents of symmetry group C_{3v} may easily be written down from the character table of the symmetry group. The results are [4,5,13,25,26]:

$$P^{(1)} = \frac{1}{6}\left(e + C_3 + C_3^{-1} + \sigma_1 + \sigma_2 + \sigma_3\right) \quad (2)$$

$$P^{(2)} = \frac{1}{6}\left(e + C_3 + C_3^{-1} - \sigma_1 - \sigma_2 - \sigma_3\right) \quad (3)$$

$$P^{(3)} = \frac{1}{3}\left(2e - C_3 - C_3^{-1}\right) \quad (4)$$

With the aid of the group table, it may easily be seen that $P^{(i)}P^{(i)} = P^{(i)}$ for $i = \{1, 2, 3\}$. Furthermore, the orthogonality property also holds, i.e. $P^{(i)}P^{(j)} = 0$ if $i \neq j$.

3. Basis vectors of the triangular space grid

Let us assume the triangular double-layer grid in Fig. 1(a) experiences small transverse motions as it undergoes free vibration. In a previous study [13], group theory was applied to the free vibration of the grids in Fig. 1, but without decomposition of their degenerate subspaces. Here, and considering only the triangular grid, we will summarise the key results of that study, to highlight the need for a further decomposition of subspace $S^{(3)}$, and to serve as a starting point from which the decomposition of subspace $S^{(3)}$ will proceed. It is important to know what subspace $S^{(3)}$ looks like before decomposition, in order to appreciate the benefit of the decomposition that will follow.

The inertia of the grid is modelled as concentrated masses $\{m_1, m_2, \ldots, m_{13}\}$ at the unsupported nodes $\{1, 2, \ldots, 13\}$ of the grid, with the vertical displacements of these masses being $\{v_1, v_2, \ldots, v_{13}\}$ respectively. The vibrating system therefore has $n = 13$ degrees of freedom $\{v_1, v_2, \ldots, v_{13}\}$. A conventional lumped-parameter vibration analysis of this system would lead to a 13×13 determinant, the vanishing condition of which results in a 13th-degree characteristic polynomial equation. Solution of the characteristic equation yields 13 eigenvalues (hence natural frequencies of the system), allowing the 13 modes of vibration to be determined. Although this is a relatively small problem, a considerable amount of effort is still required to evaluate the dynamic characteristics of the system (frequencies and modes of vibration).

On the other hand, group theory decomposes the 13×13 system matrix into a number of $r\times r$ independent matrices ($r \ll n$), which can be separately solved for all eigenvalues. This separation is achieved by applying idempotents $P^{(1)}$, $P^{(2)}$ and $P^{(3)}$, in turn, upon each of the 13 degrees of freedoms of the system, thus creating three independent subspaces $S^{(1)}$, $S^{(2)}$ and $S^{(3)}$ of the problem. When the first idempotent $P^{(1)}$ is applied (as an operator) upon $\{v_1, v_2, \ldots, v_{13}\}$, we obtain 13 *symmetry-adapted freedoms*, but not all of them are independent. We may select a set of r_1 independent symmetry-adapted freedoms as the *basis vectors* $\Phi_i^{(1)}$ ($i = 1, 2, \ldots, r_1$) of subspace $S^{(1)}$. Repeating the process using idempotents $P^{(2)}$ and $P^{(3)}$ generates the r_2 basis vectors of subspaces $S^{(2)}$ and the r_3 basis vectors of subspace $S^{(3)}$, respectively. Details may be seen in the already-mentioned previous study [13]. The results are as follows:

Subspace $S^{(1)}$

$$\Phi_1^{(1)} = v_1 + v_2 + v_3 + v_5 + v_6 + v_7 \quad (5)$$

$$\Phi_2^{(1)} = v_4 \quad (6)$$

$$\Phi_3^{(1)} = v_8 + v_{11} + v_{13} \quad (7)$$

$$\Phi_4^{(1)} = v_9 + v_{10} + v_{12} \quad (8)$$

Subspace $S^{(2)}$

$$\Phi_1^{(2)} = v_1 - v_2 - v_3 + v_5 + v_6 - v_7 \quad (9)$$

Subspace $S^{(3)}$

$$\Phi_1^{(3)} = 2v_1 - v_5 - v_6 \quad (10)$$

$$\Phi_2^{(3)} = 2v_5 - v_1 - v_6 \quad (11)$$

$$\Phi_3^{(3)} = 2v_2 - v_3 - v_7 \quad (12)$$

$$\Phi_4^{(3)} = 2v_3 - v_2 - v_7 \quad (13)$$

$$\Phi_5^{(3)} = 2v_8 - v_{11} - v_{13} \quad (14)$$

$$\Phi_6^{(3)} = 2v_{11} - v_8 - v_{13} \quad (15)$$

$$\Phi_7^{(3)} = 2v_9 - v_{10} - v_{12} \quad (16)$$

$$\Phi_8^{(3)} = 2v_{10} - v_9 - v_{12} \quad (17)$$

Clearly, subspaces $S^{(1)}$, $S^{(2)}$ and $S^{(3)}$ are 4-dimensional ($r_1 = 4$), 1-dimensional ($r_2 = 1$) and 8-dimensional ($r_3 = 8$) respectively. Thus, these subspaces will have 4, 1 and 8 modes of vibration respectively. Instead of solving a conventional 13×13 matrix system of unknowns, the above group-theoretic decomposition requires us to solve, independently of each other, three smaller matrix systems of dimensions 4×4, 1×1 and 8×8 respectively, which simplifies the problem. However, if the 8×8 matrix system of subspace $S^{(3)}$ (associated with doubly repeating solutions) can be decomposed further into two 4×4 subsystems that are *identical*, that would reduce the overall computations even further. It would mean that only three matrix systems of dimensions $\{4\times 4;\ 1\times 1;\ 4\times 4\}$ would need to be tackled in order to generate all 13 eigenvalues of the original problem, noting that the two 4×4 sub-matrices of subspace $S^{(3)}$ would be identical (therefore only one would need to be considered). This promises a significant further reduction in computational effort, and provides strong motivation for seeking a further decomposition of subspace $S^{(3)}$.

If the basis vectors of each subspace are plotted as shown in Fig. 2,

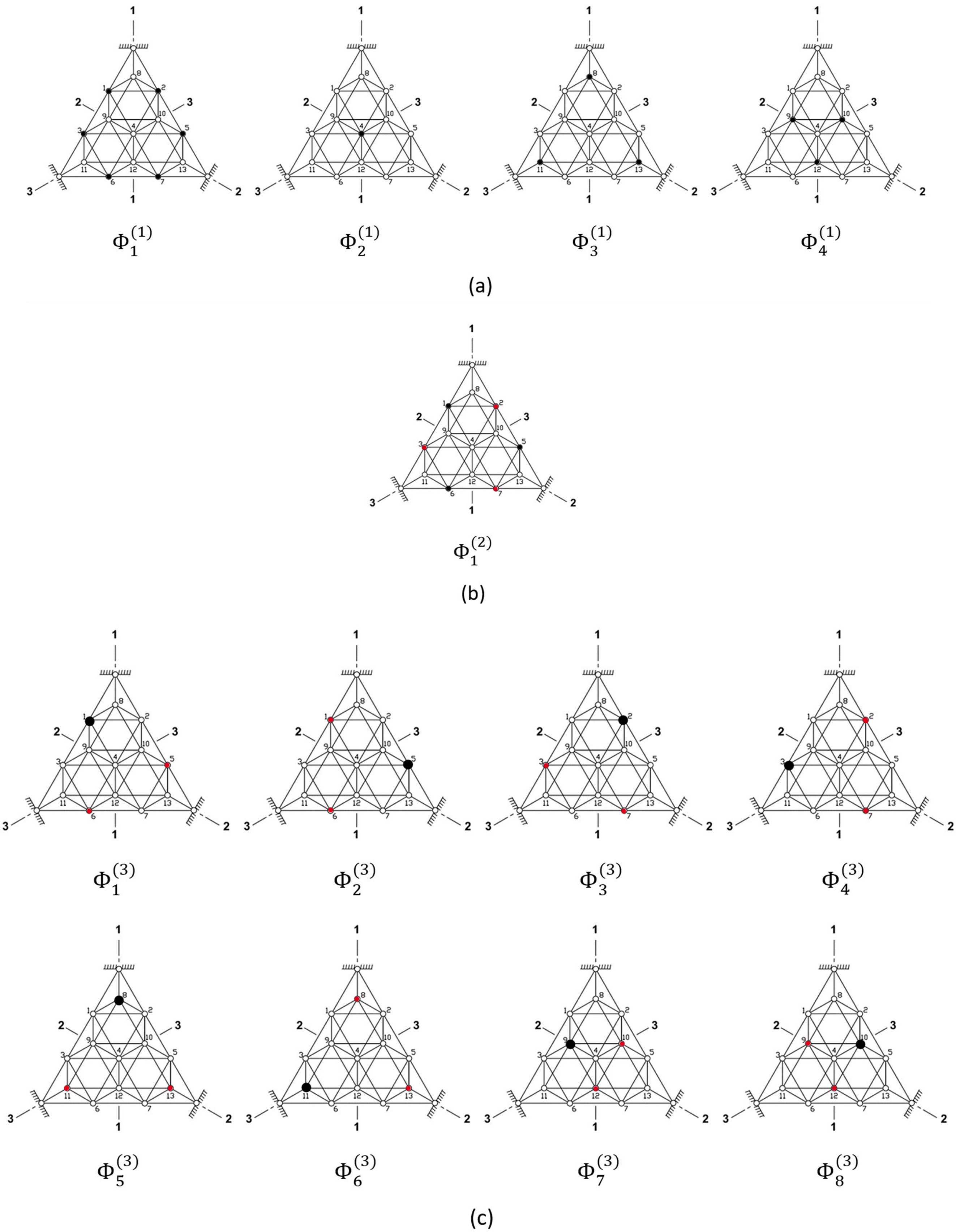

Fig. 2. Basis-vector plots of the subspaces of the triangular space grid: (a) subspace $S^{(1)}$; (b) subspace $S^{(2)}$; (c) subspace $S^{(3)}$ [13].

the symmetries of the subspaces become more evident. The plotted values are the coefficients of the v terms in Eqs. (5–17). Black dots denote positive coefficients of v (downward displacement of the node), while red dots denote negative coefficients of v (upward displacement of the node), the diameter of the dot being proportional to the coefficient. The four modes of subspace $S^{(1)}$ have the full C_{3v} symmetry of the triangular grid, while the one mode of subspace $S^{(2)}$ is antisymmetric about all three axes (1 – 1, 2 –2 and 3 – 3). On the other hand, subspace $S^{(3)}$ has a mix of symmetries (four modes have one axis of reflection, while the other four appear to have no symmetry properties).

In summary, we have noted that the irreducible representation associated with subspace $S^{(3)}$ is a 2-dimensional representation of the symmetry group C_{3v} [3,4,12,21]. This implies that the eigenvalues in this subspace will be doubly repeating. Thus, for the case of the triangular grid that we have studied, this subspace will be expected to have only four distinct eigenvalues. However, unless a way can be found of decomposing subspace $S^{(3)}$ further, an 8-dimensional eigenvalue problem (leading to an 8th-degree characteristic equation) will still need to be solved in order to arrive at the four doubly repeating solutions. This requires considerable computational effort. In the next section, a special pair of operators is proposed for the further decomposition of subspace $S^{(3)}$, to reduce computational effort.

4. Special operators for subspace $S^{(3)}$ of symmetry group C_{3v}

For problems involving symmetry group C_{3v}, we seek two operators $P^{(3,1)}$ and $P^{(3,2)}$ that are able to subdivide the degenerate subspace $S^{(3)}$ into two smaller subspaces $S^{(3,1)}$ and $S^{(3,2)}$ spanned by linear combinations of the basis vectors of subspace $S^{(3)}$, such that the basis vectors of subspace $S^{(3,1)}$ are orthogonal to those of subspace $S^{(3,2)}$. This would then allow subspaces $S^{(3,1)}$ and $S^{(3,2)}$ to be treated separately. We require these operators to satisfy the following four conditions:

$$P^{(3,1)} + P^{(3,2)} = P^{(3)} \tag{18}$$

$$P^{(3,1)}P^{(3,1)} = P^{(3,1)} \tag{19}$$

$$P^{(3,2)}P^{(3,2)} = P^{(3,2)} \tag{20}$$

$$P^{(3,1)}P^{(3,2)} = 0 \tag{21}$$

The first condition is the requirement that the sum of the two special operators must equal the idempotent $P^{(3)}$ as given by Eq. (4). The second and third conditions require the two special operators to have the property $P^{(i)}P^{(i)} = P^{(i)}$ common to all idempotents. The last condition requires the two special operators to be orthogonal to each other, to ensure the orthogonality of the basis-vector sets of subspaces $S^{(3,1)}$ and $S^{(3,2)}$.

To preserve the rotational symmetries of the parent idempotent $P^{(3)}$ (see Eq. (4)), let each of the sought operators $P^{(3,1)}$ and $P^{(3,2)}$ comprise half of $P^{(3)}$ and a linear combination of reflection elements $\{\sigma_1, \sigma_2, \sigma_3\}$ that is of equal magnitude but of opposite sign (i.e. the linear combination of $\{\sigma_1, \sigma_2, \sigma_3\}$ in $P^{(3,1)}$ must be the negative of that in $P^{(3,2)}$ so that the sum of the two linear combinations is zero). Such a pair of operators would automatically satisfy Eq. (18). The following expressions for $P^{(3,1)}$ and $P^{(3,2)}$ satisfy this requirement (hence also Eq. (18)):

$$P^{(3,1)} = \frac{1}{6}\left(2e - C_3 - C_3^{-1} - \sigma_1 - \sigma_2 + 2\sigma_3\right) \tag{22}$$

$$P^{(3,2)} = \frac{1}{6}\left(2e - C_3 - C_3^{-1} + \sigma_1 + \sigma_2 - 2\sigma_3\right) \tag{23}$$

Let us start by checking if $P^{(3,1)}$ and $P^{(3,2)}$ are orthogonal to each other (i.e. if condition (21) is satisfied). Using the above expressions for $P^{(3,1)}$ and $P^{(3,2)}$, we may write:-

$$\begin{aligned}P^{(3,1)}P^{(3,2)} &= \frac{1}{36}\left(2e - C_3 - C_3^{-1} - \sigma_1 - \sigma_2 + 2\sigma_3\right)\left(2e - C_3 - C_3^{-1} + \sigma_1 + \sigma_2 - 2\sigma_3\right)\\ &= \frac{1}{36}M\end{aligned} \tag{24}$$

where

$$M = \left(2e - C_3 - C_3^{-1} - \sigma_1 - \sigma_2 + 2\sigma_3\right)\left(2e - C_3 - C_3^{-1} + \sigma_1 + \sigma_2 - 2\sigma_3\right) \tag{25}$$

The right-hand side of Eq. (25), evaluated with the aid of the group table for symmetry group C_{3v} (Table 1), yields the following result, where line ($i = 1, 2, \ldots, 6$) is generated by multiplying term i of the first brackets by all terms of the second brackets:

$$4e - 2C_3 - 2C_3^{-1} + 2\sigma_1 + 2\sigma_2 - 4\sigma_3 \quad \text{(i)}$$

$$-2C_3 + C_3^{-1} + e - \sigma_2 - \sigma_3 + 2\sigma_1 \quad \text{(ii)}$$

$$-2C_3^{-1} + e + C_3 - \sigma_3 - \sigma_1 + 2\sigma_2 \quad \text{(iii)}$$

$$-2\sigma_1 + \sigma_3 + \sigma_2 - e - C_3^{-1} + 2C_3 \quad \text{(iv)}$$

$$-2\sigma_2 + \sigma_1 + \sigma_3 - C_3 - e + 2C_3^{-1} \quad \text{(v)}$$

$$4\sigma_3 - 2\sigma_2 - 2\sigma_1 + 2C_3^{-1} + 2C_3 - 4e \quad \text{(vi)}$$

The sum of expressions (i) – (vi) is zero. Thus M is zero, proving the orthogonality of $P^{(3,1)}$ and $P^{(3,2)}$. Similarly multiplying out $P^{(3,1)}P^{(3,1)}$ and $P^{(3,2)}P^{(3,2)}$ with the aid of the group table, we find that the operators $P^{(3,1)}$ and $P^{(3,2)}$ also satisfy Eqs. (19) and (20). Therefore relations (22) and (23), proposed here for the first time, are the sought algebraic operators for the further decomposition of subspace $S^{(3)}$. They have all the properties of idempotents, so they may be regarded as the idempotents of the semi-subspaces $S^{(3,1)}$ and $S^{(3,2)}$.

Let us assume the parent subspace $S^{(3)}$ is of dimension r_3 (this is always an even integer) before it is decomposed. When applied upon the normal variables of a problem, operators $P^{(3,1)}$ and $P^{(3,2)}$ automatically generate the $r_3/2$ basis vectors of subspace $S^{(3,1)}$ and the $r_3/2$ basis vectors of subspace $S^{(3,2)}$, respectively, thus decomposing subspace $S^{(3)}$ into two subspaces that are each half the size of subspace $S^{(3)}$.

It must be noted that *the operators given by* Eqs. (22) and (23) *are very general, being applicable to any problem that has* C_{3v} *symmetry*, whether this is a group of atoms making up a molecule (in chemistry), a crystal structure (in crystallography), a finite element (in computational engineering mechanics), or a structural-engineering system (cable net, space grid, plate, shell, plane frame, etc). In the next section, the two operators will be applied to the further decomposition of subspace $S^{(3)}$ of the triangular space grid of Fig. 1(a).

5. Application of operators to the triangular space grid

By reference to Fig. 1(a), and with rotations being about the vertical axis through node 4 and reflections being in the vertical planes 1 – 1, 2 –2 and 3 –3 as shown, the freedom v_1 is moved to freedoms $\{v_1, v_5, v_6, v_2, v_3, v_7\}$ by the symmetry operations $\{e, C_3, C_3^{-1}, \sigma_1, \sigma_2, \sigma_3\}$ respectively. Therefore, applying the operator $P^{(3,1)}$ (Eq. (22)) on v_1, we obtain

$$\begin{aligned}P^{(3,1)}v_1 &= \frac{1}{6}\left(2e - C_3 - C_3^{-1} - \sigma_1 - \sigma_2 + 2\sigma_3\right)v_1\\ &= \frac{1}{6}(2v_1 - v_5 - v_6 - v_2 - v_3 + 2v_7)\end{aligned} \tag{26a}$$

Applying the operator $P^{(3,1)}$ on the rest of the freedoms ($v_2, v_3, \ldots,$

v_{13}) of the triangular grid in a similar way, we obtain the following results:

$$P^{(3,1)}v_2 = \frac{1}{6}(2v_2 - v_7 - v_3 - v_1 - v_6 + 2v_5) \tag{26b}$$

$$P^{(3,1)}v_3 = \frac{1}{6}(2v_3 - v_2 - v_7 - v_5 - v_1 + 2v_6) = -\left[P^{(3,1)}v_1 + P^{(3,1)}v_2\right] \tag{26c}$$

$$P^{(3,1)}v_4 = 0 \tag{26d}$$

$$P^{(3,1)}v_5 = \frac{1}{6}(2v_5 - v_6 - v_1 - v_3 - v_7 + 2v_2) = P^{(3,1)}v_2 \tag{26e}$$

$$P^{(3,1)}v_6 = \frac{1}{6}(2v_6 - v_1 - v_5 - v_7 - v_2 + 2v_3) = P^{(3,1)}v_3 \tag{26 f}$$

$$P^{(3,1)}v_7 = \frac{1}{6}(2v_7 - v_3 - v_2 - v_6 - v_5 + 2v_1) = P^{(3,1)}v_1 \tag{26 g}$$

$$P^{(3,1)}v_8 = \frac{1}{6}(2v_8 - v_{13} - v_{11} - v_8 - v_{11} + 2v_{13}) = \frac{1}{6}(v_8 + v_{13} - 2v_{11}) \tag{26 h}$$

$$P^{(3,1)}v_9 = \frac{1}{6}(2v_9 - v_{10} - v_{12} - v_{10} - v_9 + 2v_{12}) = \frac{1}{6}(v_9 + v_{12} - 2v_{10}) \tag{26i}$$

$$\begin{aligned} P^{(3,1)}v_{10} &= \frac{1}{6}(2v_{10} - v_{12} - v_9 - v_9 - v_{12} + 2v_{10}) = \frac{2}{6}(2v_{10} - v_9 - v_{12}) \\ &= -2P^{(3,1)}v_9 \end{aligned} \tag{26j}$$

$$\begin{aligned} P^{(3,1)}v_{11} &= \frac{1}{6}(2v_{11} - v_8 - v_{13} - v_{13} - v_8 + 2v_{11}) = \frac{2}{6}(2v_{11} - v_8 - v_{13}) \\ &= -2P^{(3,1)}v_8 \end{aligned} \tag{26k}$$

$$\begin{aligned} P^{(3,1)}v_{12} &= \frac{1}{6}(2v_{12} - v_9 - v_{10} - v_{12} - v_{10} + 2v_9) = \frac{1}{6}(v_9 + v_{12} - 2v_{10}) \\ &= -\frac{1}{2}P^{(3,1)}v_{10} \end{aligned} \tag{26 l}$$

$$\begin{aligned} P^{(3,1)}v_{13} &= \frac{1}{6}(2v_{13} - v_{11} - v_8 - v_{11} - v_{13} + 2v_8) = \frac{1}{6}(v_8 + v_{13} - 2v_{11}) \\ &= P^{(3,1)}v_8 \end{aligned} \tag{26 m}$$

Of the 13 symmetry-adapted freedoms, i.e. $P^{(3,1)}v_i$ ($i = 1, 2, \dots, 13$), only four are independent. Choosing the symmetry-adapted freedoms in Eqs. (26a), (26b), (26h) and (26i) as the independent ones, and dropping the scalar multipliers in front of the brackets, the four basis vectors for subspace $S^{(3,1)}$ (generated by operator $P^{(3,1)}$) may be taken as follows:

Subspace $S^{(3,1)}$

$$\Phi_1^{(3,1)} = 2v_1 - v_5 - v_6 - v_2 - v_3 + 2v_7 \tag{27a}$$

$$\Phi_2^{(3,1)} = 2v_2 - v_7 - v_3 - v_1 - v_6 + 2v_5 \tag{27b}$$

$$\Phi_3^{(3,1)} = v_8 + v_{13} - 2v_{11} \tag{27c}$$

$$\Phi_4^{(3,1)} = v_9 + v_{12} - 2v_{10} \tag{27d}$$

Similarly, applying the operator $P^{(3,2)}$ on the 13 freedoms of the grid ($v_1, v_2, v_3, \dots, v_{13}$), we find that there are also only four independent symmetry-adapted freedoms, allowing us to adopt the following as the four basis vectors for subspace $S^{(3,2)}$ (generated by operator $P^{(3,2)}$):

Subspace $S^{(3,2)}$

$$\Phi_1^{(3,2)} = 2v_1 - v_5 - v_6 + v_2 + v_3 - 2v_7 \tag{28a}$$

$$\Phi_2^{(3,2)} = 2v_2 - v_7 - v_3 + v_1 + v_6 - 2v_5 \tag{28b}$$

$$\Phi_3^{(3,2)} = v_8 - v_{13} \tag{28c}$$

$$\Phi_4^{(3,2)} = v_9 - v_{12} \tag{28d}$$

The basis vectors of subspaces $S^{(3,1)}$ and $S^{(3,2)}$ are plotted in Fig. 3. These subspaces have distinct symmetry properties, as is clearly evident from the plots: all the basis-vector plots of subspace $S^{(3,1)}$ are *symmetric* about the vertical plane 3 − 3, while all the basis-vector plots of subspace $S^{(3,2)}$ are *antisymmetric* about the vertical plane 3 − 3. We therefore expect the vibration modes of subspaces $S^{(3,1)}$ and $S^{(3,2)}$ to exhibit similar patterns of symmetry properties. Thus, apart from simplifying the computation of actual frequencies and mode shapes of the parent subspace $S^{(3)}$, the operators $P^{(3,1)}$ and $P^{(3,2)}$ also *untangle* the symmetries of the subspace, separating them into C_{1v}-symmetric modes (i.e. modes with one plane of *symmetry*) and C_1-symmetric modes (i.e. modes with one plane of *antisymmetry*).

For each mode in subspace $S^{(3,1)}$, there will be a corresponding mode in subspace $S^{(3,2)}$ that has an identical natural frequency (explaining the phenomenon of doubly-repeating frequencies associated with the parent subspace $S^{(3)}$). However, the basis-vector sets of subspace $S^{(3,1)}$ and subspace $S^{(3,2)}$ are orthogonal to each other, i.e. $\Phi_i^{(3,1)}\Phi_j^{(3,2)} = 0$ for any $i = \{1, 2, 3, 4\}$ and any $j = \{1, 2, 3, 4\}$. For example, writing the coefficients of basis vectors $\Phi_1^{(3,1)}$, $\Phi_1^{(3,2)}$ and $\Phi_2^{(3,2)}$ (see Eqs. (27a), (28a) and (28b)) as $B_1^{(3,1)}$, $B_1^{(3,2)}$ and $B_2^{(3,2)}$ respectively, we obtain:-

$$\left\{B_1^{(3,1)}\right\}^T\left\{B_1^{(3,2)}\right\} = \begin{matrix}\{2 & -1 & -1 & 0 & -1 & -1 & 2 & 0 & \cdots\} \\ \{2 & 1 & 1 & 0 & -1 & -1 & -2 & 0 & \cdots\}^T\end{matrix} = 0$$

$$\left\{B_1^{(3,1)}\right\}^T\left\{B_2^{(3,2)}\right\} = \begin{matrix}\{2 & -1 & -1 & 0 & -1 & -1 & 2 & 0 & \cdots\} \\ \{1 & 2 & -1 & 0 & -2 & 1 & -1 & 0 & \cdots\}^T\end{matrix} = 0$$

showing that (i) $\Phi_1^{(3,1)}$ is orthogonal to $\Phi_1^{(3,2)}$, (ii) $\Phi_1^{(3,1)}$ is orthogonal to $\Phi_2^{(3,2)}$, and so forth.

In summary, the new operators $P^{(3,1)}$ and $P^{(3,2)}$ have allowed us to successfully decompose the 8-dimensional degenerate subspace of the triangular space grid (i.e. subspace $S^{(3)}$) into two 4-dimensional subspaces $S^{(3,1)}$ and $S^{(3,2)}$ which will have identical sets of eigenvalues but orthogonal sets of modes. This significantly reduces computational effort, as only one of these two subspaces needs to be considered in computations, to give all 8 solutions of subspace $S^{(3)}$.

6. Validation of operators

6.1. Vibration of a spring-mass system

To validate the proposed operators for the further decomposition of subspace $S^{(3)}$ of problems with C_{3v} symmetry, a spring-mass dynamic model with 3 degrees of freedom (d.o.f.) $\{u_1, u_2, u_3\}$ representing the rectilinear motions of 3 masses $\{m_1, m_2, m_3\}$, each of magnitude m as shown in Fig. 4(a), was considered. Each mass is connected to a rigid support by a spring of stiffness k_1, and to the other two masses by springs of stiffness k_2, as shown in Fig. 4(a). The C_{3v} symmetry of the configuration becomes clearer if it is re-drawn as shown in Fig. 4(b). Clearly the two systems (the one in Fig. 4(a), and the one in Fig. 4(b)) are dynamically equivalent.

This example was considered in a previous study of the first author [26], where natural frequencies of vibration for all subspaces of the problem were computed, but without the further decomposition of

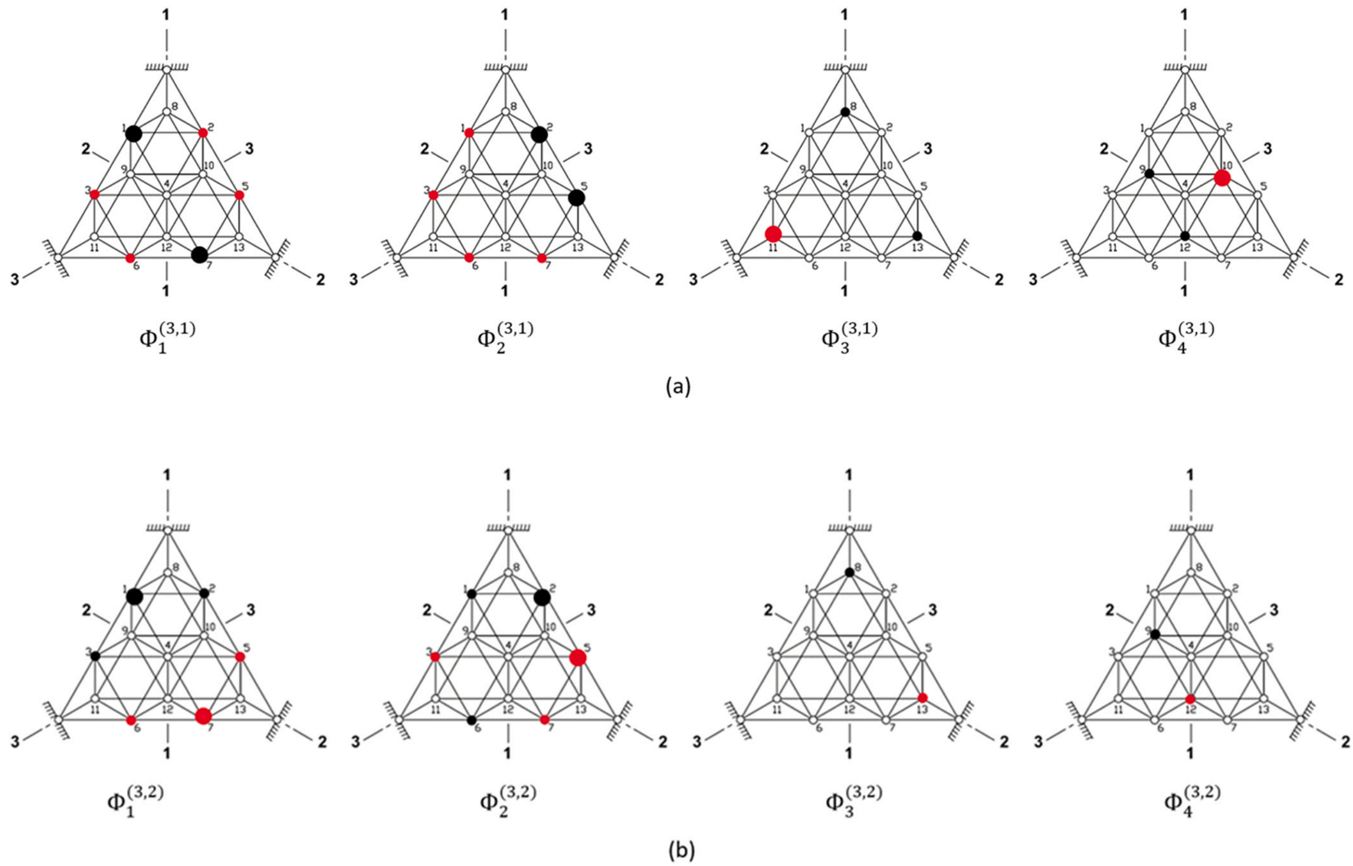

Fig. 3. Decomposition of subspace $S^{(3)}$ of the triangular space grid: (a) basis-vector plots of subspace $S^{(3,1)}$; (b) basis-vector plots of subspace $S^{(3,2)}$.

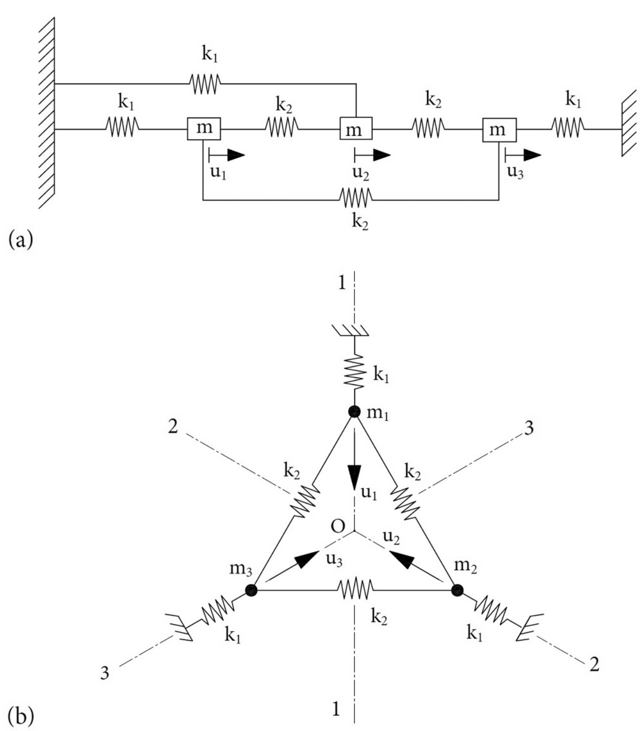

Fig. 4. A 3 d.o.f. spring–mass system: (a) actual configuration; (b) transformed configuration with C_{3v} symmetry [26].

subspace $S^{(3)}$ proposed here. By applying idempotents $P^{(1)}$, $P^{(2)}$ and $P^{(3)}$ (as given by Eqs. (2) – (4)) on the freedoms $\{u_1,\ u_2,\ u_3\}$ of the transformed configuration (Fig. 4(b)), it was found that subspace $S^{(1)}$ is 1-dimensional (i.e. it has only one basis vector), subspace $S^{(2)}$ is a null subspace, and subspace $S^{(3)}$ is 2-dimensional. In that study, subspace $S^{(3)}$ was indeed shown to have doubly repeating natural frequencies, but these frequencies were computed using the basis vectors of the 2-dimensional subspace $S^{(3)}$. With $\omega^2 = \lambda$ denoting an eigenvalue of the system, where ω is a natural circular frequency of the system, the two eigenvalues for subspace $S^{(3)}$ were obtained as follows:

$$\left(\omega^2\right)_1^{(3)} = \left(\omega^2\right)_2^{(3)} = \frac{k_1 + 3k_2}{m} \tag{29}$$

As a way of validating the newly proposed procedure for the automatic decomposition of subspace $S^{(3)}$, the operators $P^{(3,1)}$ and $P^{(3,2)}$ (Eqs. (22) and (23)) will be applied to the spring-mass system in Fig. 4, to see if the same eigenvalues (as given by Eq. (29)) can be obtained.

By reference to the configuration in Fig. 4(b) (this has 3-fold rotational symmetry about the centre of symmetry O, and three reflection planes $1-1$, $2-2$ and $3-3$ as shown), we apply operator $P^{(3,1)}$ first, to the freedoms $\{u_1,\ u_2,\ u_3\}$ in turn, to obtain symmetry-adapted freedoms as follows:

$$P^{(3,1)}u_1 = \tfrac{1}{6}\left(2e - C_3 - C_3^{-1} - \sigma_1 - \sigma_2 + 2\sigma_3\right)u_1$$

$$= \frac{1}{6}(2u_1 - u_2 - u_3 - u_1 - u_3 + 2u_2) = \frac{1}{6}(u_1 + u_2 - 2u_3) \tag{30a}$$

$$P^{(3,1)}u_2 = \frac{1}{6}\left(2e - C_3 - C_3^{-1} - \sigma_1 - \sigma_2 + 2\sigma_3\right)u_2$$

$$= \frac{1}{6}(2u_2 - u_3 - u_1 - u_3 - u_2 + 2u_1) = \frac{1}{6}(u_1 + u_2 - 2u_3) = P^{(3,1)}u_1 \tag{30b}$$

$$P^{(3,1)}u_3 = \frac{1}{6}\left(2e - C_3 - C_3^{-1} - \sigma_1 - \sigma_2 + 2\sigma_3\right)u_3$$

$$= \frac{1}{6}(2u_3 - u_1 - u_2 - u_2 - u_1 + 2u_3) = -\frac{2}{6}(u_1 + u_2 - 2u_3) = -2P^{(3,1)}u_1 \quad (30c)$$

Thus, subspace $S^{(3,1)}$ has only one independent symmetry-adapted freedom. The basis vector of subspace $S^{(3,1)}$ may be taken as

$$\Phi^{(3,1)} = u_1 + u_2 - 2u_3 = \{1 \quad 1 \quad -2\}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \{B^{(3,1)}\}^T\{U\} \quad (31)$$

where

$$\{B^{(3,1)}\} = \begin{Bmatrix} 1 \\ 1 \\ -2 \end{Bmatrix}; \{U\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad (32a, b)$$

Similarly, applying operator $P^{(3,2)}$ to the three freedoms $\{u_1, u_2, u_3\}$ in turn, we obtain symmetry-adapted freedoms as follows:

$$P^{(3,2)}u_1 = \tfrac{1}{6}\left(2e - C_3 - C_3^{-1} + \sigma_1 + \sigma_2 - 2\sigma_3\right)u_1$$

$$= \frac{1}{6}(2u_1 - u_2 - u_3 + u_1 + u_3 - 2u_2) = \frac{1}{6}(3u_1 - 3u_2) \quad (33a)$$

$$P^{(3,2)}u_2 = \tfrac{1}{6}\left(2e - C_3 - C_3^{-1} + \sigma_1 + \sigma_2 - 2\sigma_3\right)u_2$$

$$= \frac{1}{6}(2u_2 - u_3 - u_1 + u_3 + u_2 - 2u_1) = \frac{1}{6}(3u_2 - 3u_1) = -P^{(3,2)}u_1 \quad (33b)$$

$$P^{(3,2)}u_3 = \tfrac{1}{6}\left(2e - C_3 - C_3^{-1} + \sigma_1 + \sigma_2 - 2\sigma_3\right)u_3$$

$$= \frac{1}{6}(2u_3 - u_1 - u_2 + u_2 + u_1 - 2u_3) = 0 \quad (33c)$$

Thus, subspace $S^{(3,2)}$ also has only one independent symmetry-adapted freedom. The basis vector of subspace $S^{(3,2)}$ may be taken as

$$\Phi^{(3,2)} = u_1 - u_2 = \{1 \quad -1 \quad 0\}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \{B^{(3,2)}\}^T\{U\} \quad (34)$$

where

$$\{B^{(3,2)}\} = \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix}; \{U\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad (35a, b)$$

From Equations (32a) and (35a), we have

$$\{B^{(3,1)}\}^T\{B^{(3,2)}\} = \{1 \quad 1 \quad -2\}\begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix} = 1 - 1 + 0 = 0 \quad (36)$$

showing that basis vectors $\Phi^{(3,1)}$ and $\Phi^{(3,2)}$, spanning subspaces $S^{(3,1)}$ and $S^{(3,2)}$ respectively, are orthogonal to each other, as we would expect them to be (in the light of the already proven orthogonality of operators $P^{(3,1)}$ and $P^{(3,2)}$).

To obtain the symmetry-adapted stiffness matrices of subspaces $S^{(3,1)}$ and $S^{(3,2)}$, we return to the original physical system as shown in Fig. 4(a). We assign unit values of $\{u_1, u_2, u_3\}$ in the basis-vector expressions for $\Phi^{(3,1)}$ and $\Phi^{(3,2)}$ as given by Eqs. (31) and (34), then *simultaneously* apply the ensuing displacements on the masses $\{m_1, m_2, m_3\}$. The applied displacements $\{b_1, b_2, b_3\}$, on masses $\{m_1, m_2, m_3\}$ respectively, are simply given by the elements of $B^{(3,1)}$ for subspace $S^{(3,1)}$ (see Equation (32a)) and the elements of $B^{(3,2)}$ for subspace $S^{(3,2)}$ (see Equation (35a)). The application of these displacements is illustrated in Fig. 5. The displacements are considered positive when pointing towards the right.

As a result of the displacements of the masses, the springs will be stretched or compressed (as the case may be), exerting forces on the masses that tend to restore the masses to their equilibrium positions. The net restoring forces on masses $\{m_1, m_2, m_3\}$, considered positive when pointing towards the left (i.e. opposite to the direction of positive displacements), are shown in Table 2 for both the situation in Fig. 5(a) and the situation in Fig. 5(b).

By reference to Table 2, and considering subspace $S^{(3,1)}$ first, the restoring force per unit value of b is $(k_1 + 3k_2)$ at each location of mass. Therefore, the stiffness $k^{(3,1)}$ for subspace $S^{(3,1)}$ is given by

$$k^{(3,1)} = (k_1 + 3k_2) \quad (37)$$

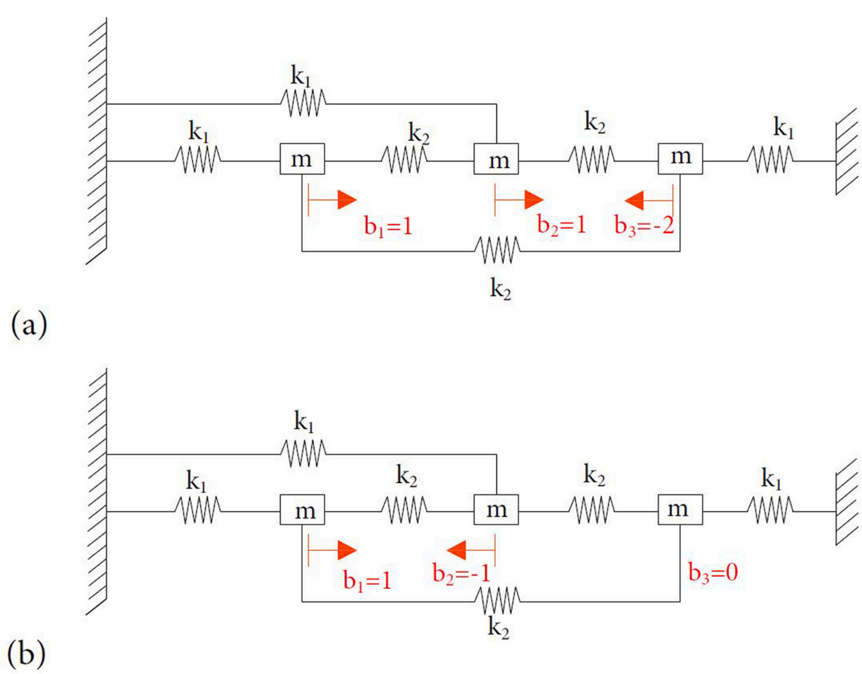

Fig. 5. Application of displacement components of basis vectors for the 3 d.o.f. spring-mass dynamic system: (a) displacements of subspace $S^{(3,1)}$; (b) displacements of subspace $S^{(3,2)}$.

Table 2
Restoring forces on masses $\{m_1, m_2, m_3\}$ due to displacements $\{b_1, b_2, b_3\}$.

Mass	Subspace $S^{(3,1)}$		Subspace $S^{(3,2)}$	
	Displacement	Restoring Force	Displacement	Restoring Force
$m_1 = m$	$b_1 = +1.0$	$(k_1 + 3k_2)$	$b_1 = +1.0$	$(k_1 + 3k_2)$
$m_2 = m$	$b_2 = +1.0$	$(k_1 + 3k_2)$	$b_2 = -1.0$	$-(k_1 + 3k_2)$
$m_3 = m$	$b_3 = -2.0$	$-2(k_1 + 3k_2)$	$b_3 = 0.0$	0.0

The mass upon which this force acts has the value m at each location of mass. Therefore, the characteristic equation for subspace $S^{(3,1)}$, given by $\left\{k^{(3,1)} - \omega^2 m = 0\right\}$, becomes

$$(k_1 + 3k_2) - \omega^2 m = 0 \tag{38}$$

where ω^2 (the square of the natural circular frequency of the system) is the one eigenvalue of the subspace. Thus, and by rearranging Eq. (38), the solution for the eigenvalue of subspace $S^{(3,1)}$ is simply given by

$$\left(\omega^2\right)^{(3,1)} = \frac{k_1 + 3k_2}{m} \tag{39}$$

Considering subspace $S^{(3,2)}$ next, the restoring force per unit value of b is also $(k_1 + 3k_2)$ at each active location of mass. Therefore, the stiffness $k^{(3,2)}$ for subspace $S^{(3,2)}$ is given by

$$k^{(3,2)} = (k_1 + 3k_2) \tag{40}$$

This leads to the characteristic equation

$$(k_1 + 3k_2) - \omega^2 m = 0 \tag{41}$$

for subspace $S^{(3,2)}$, with the solution

$$\left(\omega^2\right)^{(3,2)} = \frac{k_1 + 3k_2}{m} \tag{42}$$

The result for subspace $S^{(3,1)}$ is identical to that for subspace $S^{(3,2)}$ – compare Eqs. (39) and (42). Thus, through use of operators $P^{(3,1)}$ and $P^{(3,2)}$ (Eqs. (22) and (23)), we have successfully decomposed the 2-dimensional space of the degenerate subspace $S^{(3)}$ of the spring-mass system, into two independent 1-dimensional subspaces $S^{(3,1)}$ and $S^{(3,2)}$, allowing the computation of the doubly-repeating solutions of subspace $S^{(3)}$ to be performed separately and more easily within the smaller subspaces $S^{(3,1)}$ and $S^{(3,2)}$. The subspaces $S^{(3,1)}$ and $S^{(3,2)}$ yield equal eigenvalues (the repeating solutions of subspace $S^{(3)}$), but the eigenvectors of the two subspaces are orthogonal, the mode shapes being given by $B^{(3,1)}$ and $B^{(3,2)}$ (the coefficients of the respective basis vectors) – see Equations (32a) and (35a).

The results for the eigenvalues of subspace $S^{(3)}$, as obtained here (see Eqs. (39) and (42)), are exactly identical to those obtained in earlier work [26], but without the use of operators $P^{(3,1)}$ and $P^{(3,2)}$ – see the result in Eq. (29).

By validating the results obtained here against those reported in earlier work [26], we have shown that the proposed operators $P^{(3,1)}$ and $P^{(3,2)}$ can be reliably used to achieve a further decomposition of the degenerate subspace $S^{(3)}$ of problems belonging to the symmetry group C_{3v}, into two independent subspaces $S^{(3,1)}$ and $S^{(3,2)}$ that feature identical sets of eigenvalues, but orthogonal sets of eigenvectors.

6.2. In-plane buckling of a triangular frame

As additional validation, the buckling of a rigid 3-sided regular polygonal frame, under the compressive action of equal joint loads directed towards the centre of symmetry, was considered. The arrangement is shown in Fig. 6(a). For simplicity, we assume the frame has only three degrees of freedom, namely the joint rotations $\{\theta_1, \theta_2, \theta_3\}$ as shown in Fig. 6(b), and on this basis, use group theory in

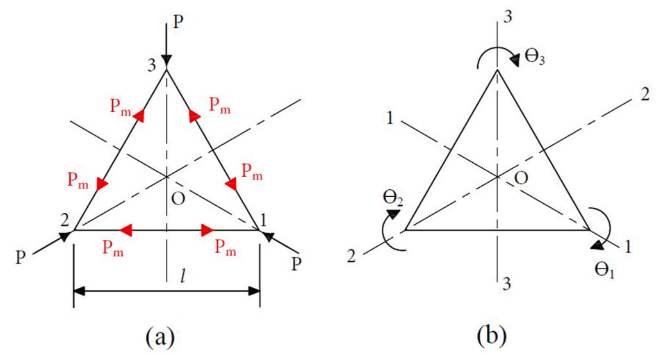

Fig. 6. C_{3v}-symmetric triangular frame subjected to point loads P directed towards centre of symmetry: (a) loading configuration; (b) rotational joint freedoms [34].

combination with the slope-deflection method of structural analysis to calculate the buckling loads (eigenvalues) and buckling modes (eigenvectors) of the frame under this loading arrangement. This problem was considered in a recent study [34], but without the use of the new operators. By applying the idempotent $P^{(3)}$ (as given by Eq. (4)) on the rotational freedoms $\{\theta_1, \theta_2, \theta_3\}$, the basis vectors for subspace $S^{(3)}$ were found to be as follows:

$$\Phi_1^{(3)} = 2\theta_1 - \theta_2 - \theta_3 \tag{43a}$$

$$\Phi_2^{(3)} = 2\theta_2 - \theta_1 - \theta_3 \tag{43b}$$

Clearly, subspace $S^{(3)}$ is 2-dimensional. In previous work [34], further decomposition of subspace $S^{(3)}$ was achieved by a search for a linear combination of the two basis vectors of subspace $S^{(3)}$ (Eqs. (43)), such that the ensuing basis vectors were orthogonal to each other, thus yielding the basis vectors of the semi-subspaces $S^{(3,1)}$ and $S^{(3,2)}$. Here, we will decompose subspace $S^{(3)}$ more systematically by using the operators in Eqs. (22) and (23).

By reference to the system of symmetry axes shown in Fig. 6(b), we first apply the operator $P^{(3,1)}$ (Eq. (22)) on the rotational freedoms $\{\theta_1, \theta_2, \theta_3\}$, to obtain:-

$$P^{(3,1)}\theta_1 = \tfrac{1}{6}\left(2e - C_3 - C_3^{-1} - \sigma_1 - \sigma_2 + 2\sigma_3\right)\theta_1$$

$$= \frac{1}{6}(2\theta_1 - \theta_2 - \theta_3 + \theta_1 + \theta_3 - 2\theta_2) = \frac{3}{6}(\theta_1 - \theta_2) \tag{44a}$$

$$P^{(3,1)}\theta_2 = \tfrac{1}{6}\left(2e - C_3 - C_3^{-1} - \sigma_1 - \sigma_2 + 2\sigma_3\right)\theta_2$$

$$= \frac{1}{6}(2\theta_2 - \theta_3 - \theta_1 + \theta_3 + \theta_2 - 2\theta_1) = -\frac{3}{6}(\theta_1 - \theta_2) = -P^{(3,1)}\theta_1 \tag{44b}$$

$$P^{(3,1)}\theta_3 = \tfrac{1}{6}\left(2e - C_3 - C_3^{-1} - \sigma_1 - \sigma_2 + 2\sigma_3\right)\theta_3$$

$$= \frac{1}{6}(2\theta_3 - \theta_1 - \theta_2 + \theta_2 + \theta_1 - 2\theta_3) = 0 \tag{44c}$$

Thus, subspace $S^{(3,1)}$ is 1-dimensional, and its basis vector may be taken as

$$\Phi^{(3,1)} = \theta_1 - \theta_2 = \{1 \quad -1 \quad 0\}\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix} = \left\{B^{(3,1)}\right\}^T\{\Theta\} \tag{45}$$

where

$$\left\{B^{(3,1)}\right\} = \begin{Bmatrix}1\\ -1\\ 0\end{Bmatrix}; \{\Theta\} = \begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix} \tag{46a, b}$$

Similarly, applying operator $P^{(3,2)}$ (Eq. (23)) on the rotations $\{\theta_1, \theta_2, \theta_3\}$, we obtain:-

$$P^{(3,2)}\theta_1 = \frac{1}{6}\left(2e - C_3 - C_3^{-1} + \sigma_1 + \sigma_2 - 2\sigma_3\right)\theta_1$$

$$= \frac{1}{6}(2\theta_1 - \theta_2 - \theta_3 - \theta_1 - \theta_3 + 2\theta_2) = \frac{1}{6}(\theta_1 + \theta_2 - 2\theta_3) \quad (47a)$$

$$P^{(3,2)}\theta_2 = \frac{1}{6}\left(2e - C_3 - C_3^{-1} + \sigma_1 + \sigma_2 - 2\sigma_3\right)\theta_2$$

$$= \frac{1}{6}(2\theta_2 - \theta_3 - \theta_1 - \theta_3 - \theta_2 + 2\theta_1) = \frac{1}{6}(\theta_1 + \theta_2 - 2\theta_3) = P^{(3,2)}\theta_1 \quad (47b)$$

$$P^{(3,2)}\theta_3 = \frac{1}{6}\left(2e - C_3 - C_3^{-1} + \sigma_1 + \sigma_2 - 2\sigma_3\right)\theta_3$$

$$= \frac{1}{6}(2\theta_3 - \theta_1 - \theta_2 - \theta_2 - \theta_1 + 2\theta_3) = -\frac{2}{6}(\theta_1 + \theta_2 - 2\theta_3) = -2P^{(3,2)}\theta_1 \quad (47c)$$

Thus, subspace $S^{(3,2)}$ is also 1-dimensional. Its basis vector may be taken as:-

$$\Phi^{(3,2)} = \theta_1 + \theta_2 - 2\theta_3 = \{1 \quad 1 \quad -2\}\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix} = \{B^{(3,2)}\}^T\{\Theta\} \quad (48)$$

where

$$\{B^{(3,2)}\} = \begin{Bmatrix}1\\ 1\\ -2\end{Bmatrix}; \{\Theta\} = \begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix} \quad (49a, b)$$

From Eqs. (46a) and (49a), we have

$$\{B^{(3,1)}\}^T\{B^{(3,2)}\} = \{1 \quad -1 \quad 0\}\begin{Bmatrix}1\\ 1\\ -2\end{Bmatrix} = 1 - 1 + 0 = 0 \quad (50)$$

showing that basis vectors $\Phi^{(3,1)}$ and $\Phi^{(3,2)}$, spanning subspaces $S^{(3,1)}$ and $S^{(3,2)}$ respectively, are orthogonal to each other, as we would expect them to be.

Plots of basis vectors $\Phi^{(3,1)}$ and $\Phi^{(3,2)}$ are shown in Fig. 7. The plot of basis vector $\Phi^{(3,1)}$ has C_{1v} symmetry (i.e. *symmetry* about axis 3 – 3), while the plot of basis vector $\Phi^{(3,2)}$ has C_1 symmetry (i.e. *antisymmetry* about axis 3 – 3). These symmetries characterise the modes of subspaces $S^{(3,1)}$ and $S^{(3,2)}$ respectively.

The basis-vector results in Eqs. (45) and (48), obtained on the basis of the new operators $P^{(3,1)}$ and $P^{(3,2)}$, are exactly the same as those that were obtained in earlier work [34] without the use of these operators. This shows yet again that, for physical problems belonging to symmetry group C_{3v}, the new operators can be relied upon to correctly decompose subspace $S^{(3)}$ into two independent semi-subspaces $S^{(3,1)}$ and $S^{(3,2)}$ whose modes are orthogonal to each other.

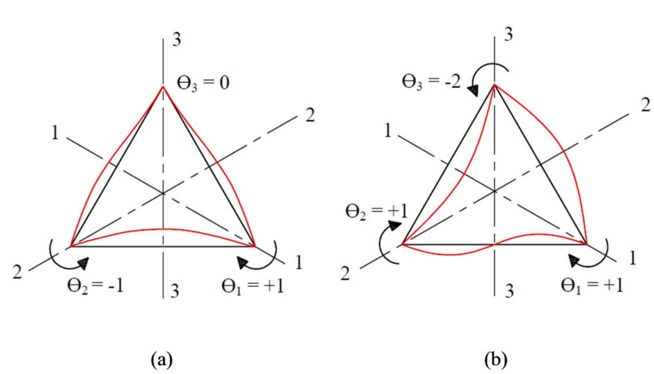

Fig. 7. Basis-vector plots of subspace $S^{(3)}$ of the C_{3v}-symmetric triangular frame: (a) $\Phi^{(3,1)}$ of subspace $S^{(3,1)}$ (C_{1v}-symmetric); (b) $\Phi^{(3,2)}$ of subspace $S^{(3,2)}$ (C_1-symmetric).

In the previous work [34], analytical results for eigenvalues and mode shapes of the triangular frame, computed from all subspaces of the problem using the group-theoretic formulation, were compared with the results obtained from a linear eigenvalue buckling analysis of the frame using the well-known finite-element programme ABAQUS. The first eight modes as computed from the finite-element analysis (FEM) are shown in Fig. 8. Comparisons between group-theoretic (GRT) and numerical (FEM) results were made for the first six modes, and are shown in Table 3. In the table, the modes of the triangular frame are presented in ascending order of natural frequencies, with their symmetries (and the subspaces to which the modes belong) indicated in column 5.

Modes $\{\lambda_1, \lambda_2\}$ in Fig. 8, corresponding to $h = 1$ and $h = 2$ in Table 3, have equal eigenvalues and belong to subspaces $S^{(3,2)}$ and $S^{(3,1)}$ respectively. Modes $\{\lambda_5, \lambda_6\}$ in Fig. 8, corresponding to $h = 5$ and $h = 6$ in Table 3, also have equal eigenvalues and belong to subspaces $S^{(3,2)}$ and $S^{(3,1)}$ respectively. These two pairs of modes belong to the degenerate parent subspace $S^{(3)}$. The new operators $P^{(3,1)}$ and $P^{(3,2)}$ permit the analytical computation of C_{1v}-symmetric $S^{(3)}$ modes (belonging to subspace $S^{(3,1)}$) to be separated from the computation of C_1-symmetric $S^{(3)}$ modes (belonging to subspace $S^{(3,2)}$).

Modes $\{\lambda_3, \lambda_4\}$ are interesting. They have equal eigenvalues, despite originating from completely different subspaces (namely, subspaces $S^{(2)}$ and $S^{(1)}$ respectively). This is purely coincidental, and stems from the stiffness peculiarities of the frame. Although we would normally associate this behaviour (equality of eigenvalues) with degenerate subspaces, different subspaces may also have modes that share the same eigenvalues under certain circumstances. A simple illustration of this behaviour is provided by the Euler buckling strut of length l but with different end conditions. If both ends of the strut are fixed, the *first* buckling mode of such a strut is *symmetric* about the midpoint of the strut and has the well-known buckling load of $4\pi^2EI/l^2$. On the other hand, if both ends of the strut are pinned (free to rotate but restrained against lateral translation), the *second* buckling mode of such a strut is *antisymmetric* about the midpoint of the strut, and also has a buckling load of $4\pi^2EI/l^2$. Clearly the two modes have different symmetry types, and belong to different subspaces, yet their eigenvalues are numerically equal.

For all subspaces of the problem, the agreement between theoretical and FEM results (see Table 3) is excellent, proving the validity of the group-theoretic formulation that forms the basis of the present work.

7. Concluding remarks

In this contribution, we have presented, for the first time, a new pair of operators for the full decomposition of the group-theoretic subspaces of structural configurations belonging to the symmetry group C_{3v}, which describes the symmetry of a 3-sided regular polygon. Within structural engineering, and in general, the group-theoretic formulation finds application in the analysis of plane frames, space frames, cable nets, space grids, lattice shells and many other types of structures, provided the structural configurations in question have symmetry properties.

Specifically, by acting on the normal variables of a C_{3v}-symmetric structural problem (such as nodal positions and degrees of freedom), the newly proposed operators generate two sets of basis vectors that are orthogonal to each other, effectively decomposing the degenerate subspace $S^{(3)}$ (associated with doubly-repeating solutions of the problem) into two independent semi-subspaces $S^{(3,1)}$ and $S^{(3,2)}$. In eigenvalue problems, the two semi-subspaces yield identical sets of eigenvalues. Modes of the same semi-subspace all have the same symmetry type: $S^{(3,1)}$ modes are *symmetric* about a vertical plane of the configuration, while $S^{(3,2)}$ modes are *antisymmetric*.

Application of these operators has been illustrated by reference to the small vertical vibrations of a double-layer triangular space grid.

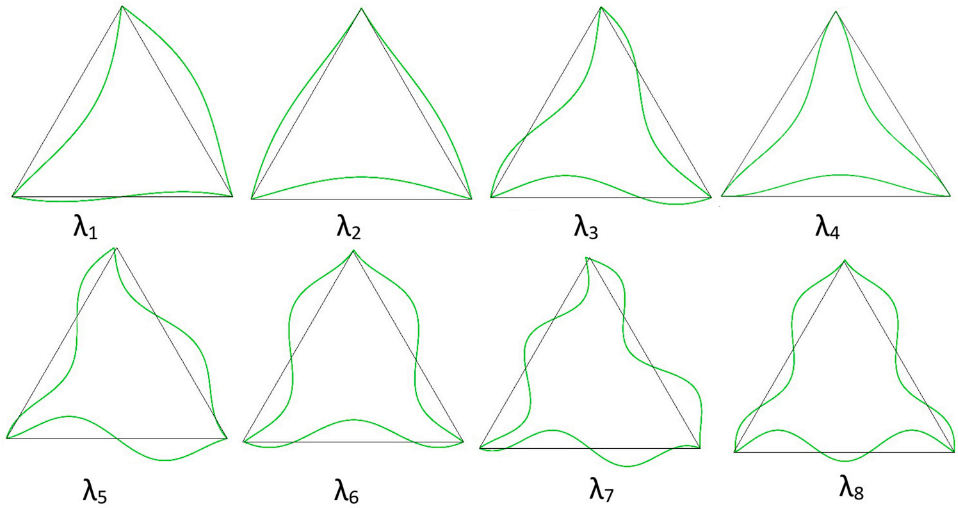

Fig. 8. First eight buckling modes of the triangular frame as yielded by FEM [34].

Table 3
Group-theoretic (GRT) versus finite-element analysis (FEM) results for the triangular frame (λ_h denotes the eigenvalue for mode h of the full system) [34].

h	λ_h (GRT)	λ_h (FEM)	% difference in λ_h values	symmetry of mode (GRT)	symmetry of mode (FEM)
1	3.857	3.813	1.2	C_1 $[S^{(3,2)}]$	C_1
2	3.857	3.813	1.2	C_{1v} $[S^{(3,1)}]$	C_{1v}
3	6.283	6.191	1.5	C_3 $[S^{(2)}]$	C_3
4	6.283	6.191	1.5	C_{3v} $[S^{(1)}]$	C_{3v}
5	8.187	8.026	2.0	C_1 $[S^{(3,2)}]$	C_1
6	8.187	8.026	2.0	C_{1v} $[S^{(3,1)}]$	C_{1v}

Their validity has been confirmed by consideration of the vibration of a spring-mass system and the in-plane buckling of a 3-sided regular polygonal frame, and the obtained eigenvalues and eigenmodes compared with existing results from the literature. These operators, which have never been presented before in such a simple form, not only simplify computations, but also "untangle" the symmetries of subspace $S^{(3)}$ for ease of study. As an extension of the approach adopted in the present work, similar operators have just been developed for studying the degenerate subspaces of the higher-order symmetry group C_{6v} describing the symmetry of hexagonal systems [35], and work on other groups is in progress.

CRediT authorship contribution statement

Alphose Zingoni: Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Formal analysis, Conceptualization. **Chisanga Kaluba:** Validation, Investigation, Formal analysis.

Declaration of Competing Interest

None.

Data availability

No data was used for the research described in the article.

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