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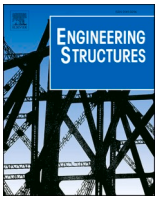

A consistent group-theoretic transformation for the block-diagonalization of structural matrices

Alphose Zingoni *, Chisanga Kaluba

Department of Civil Engineering, University of Cape Town, Rondebosch 7701, Cape Town, South Africa

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ABSTRACT

Symmetry properties of physical systems may be studied through symmetry groups. In recent times, group theory has found application in the study of various problems in structural mechanics, specifically bifurcation, buckling, kinematics and vibration. Computational simplifications are achieved by decomposing the vector space of the problem into smaller subspaces that are independent of each other. When the basis vectors of a subspace are used as the symmetry-adapted variables of that subspace, a smaller problem (associated with a matrix of smaller dimensions) automatically results. However, the same decomposition may be achieved by first obtaining the structural matrix of the system, and then transforming this into a non-overlapping block-diagonal matrix, each independent block being associated with a subspace of the problem. The advantage of this approach is its greater amenability to computer programming, but it does not always give the correct results unless a very specific procedure is followed. The purpose of this contribution is to present a consistent group-theoretic approach for the block diagonalization of structural matrices.

1. Introduction

Over the years, group theory has been extensively employed to study physical problems with symmetry properties. Within physics, the theory has been applied to various problems in quantum mechanics, and to study the physical properties of molecules and crystal structures [1–4]. In the context of engineering structures, the first application of group theory appears to have been to the problem of the buckling of symmetrical frameworks [5]. From that time onwards, engineering applications have extended to bifurcation [6–9], stability [10–12], statics [13–15], kinematics [16–20] and vibration [21–30]. A review of some of these developments has been reported [31].

Numerical implementation of group-theoretic formulations for engineering problems is still very much an ongoing area of research. Theoretical formulations that make use of concepts of symmetry groups to perform computations at certain stages of the finite-element method already exist [32–37]. For instance, in computing element matrices, group theory has been employed to split the arbitrary displacement field of a finite element into symmetry-adapted subfields, resulting in significant reductions in computational effort. Such techniques have been successfully applied not only to simple one-dimensional finite elements of the truss and beam type [32], but also to more complex three-dimensional finite elements [34–36]. Some investigators have applied group theory to the dynamic analysis of finite-element models [37]. At the level of the structure, group theory has been exploited to study the mechanics of various classes of problems, including the buckling of shells and plates [33].

Group-theoretic procedures offer the following merits: (i) reduction in computational effort as a result of the domain of the problem (i.e. its vector space) being split into independent subspaces spanned by symmetry-adapted variables; (ii) a better understanding of the mechanics of the system [24,38–40], as well as a better understanding of why certain phenomena occur. For example, in studying the vibration of symmetric structures, group theory gives us prior knowledge of the configuration or patterns of all modes of vibration before we perform any detailed calculations; it also predicts the existence of frequencies that belong to different modes but are identical in magnitude, and reveals the location of stationary points, stationary lines and stationary planes (useful in deciding the positioning of vibration-sensitive equipment). Other notable examples of the innovative use of group theory in structural mechanics have been the form-finding of tensegrity structures and study of their stability and kinematics [41–44].

It might be argued that to decompose the space of a problem into the constituent symmetry subspaces is not necessary in real computational

* Corresponding author.
E-mail address: alphose.zingoni@uct.ac.za (A. Zingoni).

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situations, since computers nowadays can handle the full problem relatively easily. While this may be a valid argument for small to medium-size problems, vector-space decomposition becomes particularly advantageous when the problem becomes very large. Such problems are characterized by a large number of unknowns, requiring substantial expenditure of computational effort to solve for these. Group theory provides a means for splitting up the large problem into a series of smaller problems, thus significantly reducing overall computational effort (the computational time expended in solving one large problem is significantly greater than the sum of the computational times of solving all subproblems of the large problem). Such decomposition of the problem spreads the computational workload over several independent subspaces, making it possible to use a number of relatively small computers in parallel with each other, and controlled by one master processor. This accelerates computations even more.

Identification of all the symmetry properties of a physical system of a problem is the first step in the practical implementation of group-theoretical computational procedures. This allows the symmetry group of the system to be determined. If the symmetry of the system is complex, and the symmetry properties are not obvious, one may have to search for these in a systematic manner; various symmetry-recognition algorithms have been proposed in this regard [45–48]. Some algorithms are intended for systems that are known to possess only cyclic symmetries [47], while other algorithms are applicable to general 3-dimensional systems in which all types of point symmetries may exist [46]. Quite often the symmetry group that takes account of all the symmetry properties of the system has one or more sub-groups of lower order, each of which may be used as the basis of a group-theoretic computational analysis. In such instances, we may very well ask which of these sub-groups (including the full group itself) is the most efficient from a computational point of view. It becomes necessary to develop suitable criteria to inform the choice of the most optimal symmetry group for a given problem [49].

While group-theoretic methods are generally effective in the analysis of physical problems with any type of symmetry, other mathematical procedures for taking into account specific types of symmetry have also been developed. Noteworthy among these are discrete Fourier methods, which have been applied to structural configurations with cyclic symmetry [50–52]. Methods based on graph theory have also been developed for the analysis of repetitive structures as well as structures have other types of regularity [53–55].

As shown through the previous work of the author and other investigators [10,13,14,21,24,26,28–30], when the basis vectors of a subspace are used as the symmetry-adapted variables of that subspace, a smaller problem (associated with a matrix of smaller dimensions) automatically results. However, the same decomposition may be achieved by first obtaining the structural matrix of the system, and then transforming this into a non-overlapping block-diagonal matrix, each independent block being associated with a subspace of the problem. The latter approach has been adopted by several investigators [6–9,11,56].

The first approach [10,13,14,21,24,26,28–30] has the advantage of being computationally more efficient; it is not necessary to assemble the structural matrix of the full system first, and subspace matrices are simply computed by noting the effects of symmetry-adapted functions on basis-vector locations (i.e. within the subspaces, symmetry-adapted functions take the place of normal functions). The disadvantage is that implementation requires more careful visualization of cause and effect, which is why it is useful to plot basis vectors first.

On the other hand, block-diagonalization is computationally more demanding, since the conventional structural matrix has to be first assembled, then converted into block-diagonal form through a suitable transformation operation. The advantage is that once the transformation matrix has been correctly set up, block-diagonalization is achieved through a series of standard matric operations, with no visualization of cause and effect being required. Thus, the second approach is more amenable to computer programming. Unfortunately, this does not always give the correct results unless a very specific procedure is followed.

The purpose of this contribution is to present a consistent group-theoretic strategy for the block diagonalization of structural matrices, towards the development of more efficient computational procedures for the solution of large eigenvalue problems of the vibration or buckling of symmetric structures (cable nets, space frames, plane grids, space grids, lattice domes, tensegrity networks, etc). The formulation is different from existing block-diagonalization procedures in two important respects: (i) a very specific convention for choosing the origin of the global coordinate system, numbering the nodes of the structural system, and choosing the positive directions of the freedoms and loads at the nodes, is defined; (ii) the transformation matrix is assembled as the square matrix connecting the array of all subspace basis vectors (arranged in a very specific manner) with the array of functions of the full vector space of the system.

Of the various block-diagonalization studies that have been reported in the literature, the work of Kangwai and co-workers [57] is particularly noteworthy, as it discussed a number of different strategies for the analysis of symmetric structures, and explained how group theory can be used to transform the stiffness matrix of a symmetric structure into block-diagonal form, with static analysis in mind, and considering examples of pin-jointed structures. The main difference between the approach of that paper and the approach adopted in the present contribution is that here (and consistent with the approach that has generally been adopted by the first author in previous studies of various vibration problems [21,24,29]), we make use of *idempotents* (linear combinations of the symmetry elements of a group as written down from the character table of the group) to derive symmetry-adapted functions from the conventional variables of the problem, whereas in the work of Kangwai and co-workers [57], use is made of the *irreducible matrix representations* of the symmetry elements as operators on conventional variables. While the later approach [57] is better able to deal with irreducible representations of dimension greater than 2, the present approach has the advantage of simplicity, as very simple algebraic expressions (rather than matrices) are used as projection operators, and character tables of point groups are readily available in the literature [3,4]. Use of idempotents is particularly effective in the case of symmetry groups where all the irreducible representations of the group happen to be 1-dimensional (such as C_{1v}, C_{2v} and D_{2h} groups). For the 2-dimensional irreducible representations of groups C_{3v}, C_{4v} and C_{6v}, techniques for the further splitting of the associated subspaces (and hence separation of doubly-repeating solutions) have been developed in previous work [21,24,26], and are currently being extended to symmetry groups of higher order.

It should also be noted that the group-theoretic block-diagonalization procedure proposed in the present work is applicable not only to the static analysis of symmetric structures [57], but also to the buckling and vibration analysis of structures, where eigenvalue equations involving symmetry-adapted matrices (flexibility, stiffness, geometric or mass matrices) need to be formulated and solved. Finally, and as already pointed out, an important distinguishing feature of the present approach is its use of a specific group-theoretic convention for node numbering and freedoms, which is central to the successful implementation of the method, and has not been reported in the literature before.

The proposed technique for deriving the transformation matrix has similarities to that described in Ref. [58], but the difference here is that we present a simpler and more consistent procedure for block-diagonalization, by first *adapting* the conventional structural matrix to a very specific group-theoretic convention for node numbering and positive directions of freedoms. This symmetry-adapted form of the conventional structural matrix, derived in a very specific manner, will herein be referred to as the *group-theoretic system matrix*; we do not make use of the concept of a *group supermatrix* [58]. Instead, we apply the transformation matrix on the group-theoretic system matrix to achieve block-diagonalization in a consistent manner. It is hoped the simplicity

of the proposed approach, and its effectiveness (as will be demonstrated in the paper), will encourage the implementation of the group-theoretic method in day-to-day engineering computations.

The structure of the rest of the paper is as follows. In Section 2, we define the specific group-theoretic convention upon which the proposed block-diagonalization transformation is based. In Section 3, the transformation matrix for group-theoretic block-diagonalization is derived, and the block-diagonalization formulation presented. In Section 4, the developed formulation is applied to a number of configurations belonging to symmetry groups C_{1v} and C_{2v}. In Section 5, the formulation is validated through a detailed consideration of a numerical example involving the small vertical vibrations of a rectangular grid. The last section contains some concluding remarks.

2. The Group-Theoretic convention for node numbering and positive directions

For a structure or structural system that has symmetry, we will adopt a special convention for the numbering of nodes (or joints) and the selection of positive directions of nodal variables, to ensure that block-diagonalization is achieved as consistently as possible, so that the procedure is reliable (always gives the correct results) and easy to implement in practical computational analysis. In the considerations of this work, we will limit our attention to structural configurations belonging to *point groups* (i.e. those groups that are associated with physical systems that have only one centre of symmetry); these are the most relevant in considerations of structural symmetry.

Consistent with previous formulations [36], the origin O of the group-theoretic co-ordinate system is taken at the centre of symmetry of the structural configuration, whose nodes may be dispersed in one dimension (linear systems), in two dimensions (planar systems) or in three dimensions (spatial systems). Reflection planes and rotation axes are then defined with respect to this origin.

To begin the numbering, node 1 is chosen on the positive side of the x axis in the case of linear systems orientated along the x axis, or in the first quadrant of the xy Cartesian coordinate system (i.e. the quadrant in which both x and y are positive) in the case of 2-D systems lying in the xy plane, or in the first octant of the xyz coordinate system (i.e. the octant in which x, y and z are all positive) in the case of 3-D systems lying in the xyz space. In the case of systems with non-orthogonal symmetry axes or non-orthogonal symmetry planes (such as triangular and hexagonal grids with C_3 and C_6 symmetries, or tetrahedral and icosahedral systems with T and I symmetries), the numbering may be commenced in any symmetry sector, which becomes the equivalent of the first quadrant of an xy Cartesian system or the first octant of an xyz Cartesian system.

In a symmetric structural configuration with a total of N nodes, we will define a nodal set as any subset of nodes that are interchanged (or permuted) by the operations of symmetry group G. All the N nodes of the system may be grouped into a finite number of nodal sets. The system may have one or more nodal sets, depending on the value of N and the arrangement of the nodes. Systems belonging to low-order symmetry groups (i.e. symmetry groups with a small number of symmetry elements), but with a large number of nodes, will typically have several (if not many) nodal sets.

Having selected Node 1, all other nodes belonging to the nodal set of Node 1 are numbered in the order generated by systematically applying the symmetry elements of G upon the position of Node 1. In this process, the symmetry operations are executed in the order in which group elements appear across the top of the character table of the group. Character tables of point groups are readily available in the traditional literature [3,4], and on many internet websites.

Once all nodes of the first nodal set have been numbered, the next node on the positive side of the x axis, or the next node in the first quadrant of a 2-dimensional system, or the next node in the first octant of a 3-dimensional system, is selected, and all the nodes of the second nodal set numbered as a continuation of the numbering of the first nodal set, and following the same sequence as applied to the first nodal set. This process is repeated for every nodal set (one nodal set after another), until all the N individual nodes of the system have been numbered consecutively from 1 up to N. If there are nodes on the centre of symmetry or on axes of symmetry of the configuration, these should be numbered last (again following the same sequence of operations), starting with nodes on the positive branch of the x axis, followed by nodes on the positive branch of the y axis, then nodes on the positive branch of the z axis, and finally ending with the node at the centre of symmetry.

With all nodes now numbered, the positive directions of nodal displacements and nodal loads (let us simply refer to these as nodal vectors) can now be assigned. While the nodes are numbered in the order generated by symmetry elements across the top of the character table (as explained above), the degrees of freedom at the nodes are prescribed differently: we first assign the positive directions of nodal vectors (displacements or loads) at Node 1 as preferred (it does not matter which directions one takes as positive), but having decided these, *the positive directions for nodal vectors at all other nodes of the system must be chosen in such a way that the overall pattern of nodal vectors (as plotted for the entire system) reflects the symmetry of the symmetry group that will be used in the analysis* (this will be either the symmetry group G describing the full symmetry of the configuration of nodes, or one of the sub-groups of G). Corresponding nodes of the different nodal sets will therefore have the same positive directions of nodal vectors. For nodal vectors that coincide with the global Cartesian directions, and taking the class of 2-dimensional systems as an example, if $\{u, v\}$ are the freedoms assigned to Node 1 in the $\{x, y\}$ directions respectively, this rule implies that the u freedoms at all other nodes will be in the x direction, and the v freedoms in the y direction. For nodal vectors that are inclined to the global Cartesian directions, it does not matter which vector at the node is taken as u or v, as long as the two vectors are symmetrically inclined relative to the axis of symmetry passing through the node.

Fig. 1(a) illustrates the group-theoretic convention for a rectangular system with C_{2v} symmetry and lying in the xy plane. This has eight nodes, and two translational degrees of freedom $\{u, v\}$ at each node. There could also be a third degree of freedom $\{w\}$ at each node in the z direction (perpendicular to the xy plane), as in the case of plane grids, space trusses and cable nets. With the origin taken at the centre of symmetry O, and the x and y axes defined as shown, the elements of symmetry group C_{2v} are: e (identity element); C_2 (rotation through an angle of π about the z axis passing through O and perpendicular to the xy plane); σ_x(reflection in the xz plane); σ_y (reflection in the yz plane). The order of elements across the top of the character table for symmetry group C_{2v} is $\{e,\ C_2,\ \sigma_x,\ \sigma_y\}$. Permutation of the nodes of the system by elements of group C_{2v} gives two nodal sets, $NS1 = \{1,\ 2,\ 3,\ 4\}$ and $NS2 = \{5,\ 6,\ 7,\ 8\}$, with positions and nodal freedoms numbered in accordance with the defined group-theoretic convention. Notice that the overall pattern of freedoms preserves the full C_{2v} symmetry of the structural configuration.

Fig. 1(b)-(e) illustrate the group-theoretic convention for various other structural systems belonging to the C_{nv} family of symmetry groups, namely the C_{4v}, C_{3v}, C_{6v} and C_{8v} groups describing the symmetries of square, triangular, hexagonal and octagonal configurations of regular shape. For these groups, the order of elements governing the numbering of nodes are as follows:

$$C_{4v}:\ e,\ C_4,\ C_4^{-1},\ C_2,\ \sigma_x,\ \sigma_y,\ \sigma_1,\ \sigma_2$$

$$C_{3v}:\ e,\ C_3,\ C_3^{-1},\ \sigma_1,\ \sigma_2,\ \sigma_3$$

$$C_{6v}:\ e,\ C_6,\ C_6^{-1},\ C_3,\ C_3^{-1},\ C_2,\ \sigma_a,\ \sigma_b,\ \sigma_c,\ \sigma_1,\ \sigma_2,\ \sigma_3$$

$$C_{8v}:\ e,\ C_8,\ C_8^{-1},\ C_4,\ C_4^{-1},\ C_8^3,\ \left(C_8^3\right)^{-1},\ C_2,\ \sigma_a,\ \sigma_b,\ \sigma_c,\ \sigma_d,\ \sigma_1,\ \sigma_2,\ \sigma_3,\ \sigma_4$$

In general, symmetry elements for C_{nv} configurations are defined

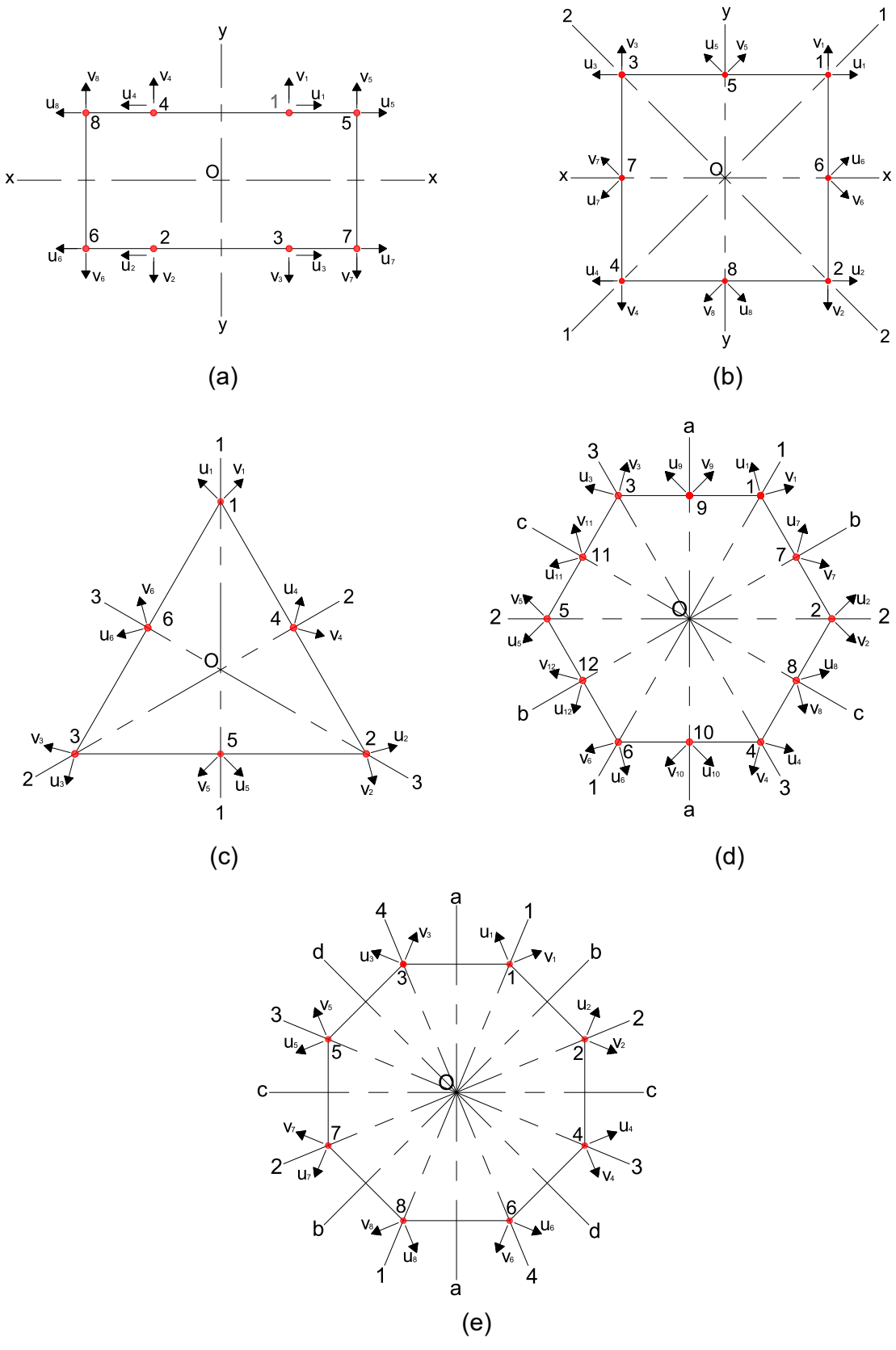

Fig. 1. Group-theoretic convention for the numbering of nodes and orientation of freedoms: (a) rectangular system with C_{2v} symmetry; (b) square system with C_{4v} symmetry; (c) equilateral-triangular system with C_{3v} symmetry; (d) regular-hexagonal system with C_{6v} symmetry; (e) regular-octagonal system with C_{8v} symmetry.

with respect to in-plane rotations about the z axis passing through the centre of symmetry (marked by O in each of the configurations) and perpendicular to the plane of the configuration, and reflections in planes containing the centre of symmetry O and perpendicular to the plane of the configuration. The rotation operation C_n denotes a clockwise rotation of $2\pi/n$, while C_n^{-1} denotes an anticlockwise rotation of $2\pi/n$. Reflection planes are marked with numbers or letters as shown in the diagrams. The rectangular configuration (Fig. 1(a)) has two Cartesian reflection planes marked $x-x$ and $y-y$, while the square configuration (Fig. 1(b)) also has the reflection planes $x-x$ and $y-y$, and additional reflection planes $1-1$ and $2-2$ coinciding with the diagonals of the configuration, as shown in the diagram.

In regular polygonal configurations with an odd number of sides n_s (see the triangular configuration of Fig. 1(c)), the n_s symmetry axes are all given numerical labels, each axis passing through a corner node and the midpoint of the opposite side. In regular polygonal configurations with an even number of sides n_s other than the square (see the hexagonal configuration of Fig. 1(d) and the octagonal configuration of Fig. 1(e)), the $n_s/2$ symmetry planes passing through corner nodes are given numerical labels ($1-1$, $2-2$, $3-3$, etc), while the $n_s/2$ symmetry planes passing through the midpoints of sides are given letter labels ($a-a$, $b-b$, $c-c$, etc). It should be noted that this labelling convention for symmetry planes of regular-polygonal configurations is only intended to make the meaning of the symmetry operations easier to understand, and is not an essential feature of the proposed group-theoretic convention as described in this section.

In all the configurations of Fig. 1(b)-(e), the freedoms $\{u,\ v\}$ at each node lie in the plane of the structure (let us assume this to be horizontal), and are inclined at 45° to the axis that passes through the node in question and the centre of symmetry O of the configuration. This means that the displacement vectors $\{u,\ v\}$ at each node are not only perpendicular to each other, but also form a pattern (when viewed collectively) that conforms to the overall symmetry of the structural configuration. The third translational freedom w (not shown), if applicable, would of course be in the vertical direction (i.e. perpendicular to the plane of the structure); the positive direction of this may be taken as either upwards at all nodes, or downwards at all nodes.

The block-diagonalization formulation that will be presented requires the matrix describing the response of the system (stiffness, flexibility, stability, dynamic, etc.) to be derived on the basis of the above system of node numbering and positive directions of nodal variables. If the formulation is simply applied to the conventional form of these matrices, the desired block-diagonalization will not be achieved. Provided the group-theoretic convention is adopted, the resulting system matrix (stiffness, flexibility, stability, dynamic, etc.) will exhibit properties that automatically reflect the physical symmetry of the structural configuration. We will refer to such a matrix as the *group-theoretic system matrix*, to distinguish it from the conventional system matrix.

Conversely, the computation of the elements of the system matrix of a symmetric configuration is simplified if the group-theoretic convention is adopted for the numbering of the nodes and the assignment of positive directions of nodal variables (i.e. calculating the group-theoretic system matrix becomes much quicker than calculating the conventional system matrix).

In algebraic computations (where the stiffness or flexibility coefficients of a system matrix are represented by algebraic symbols for greater generality), adopting the group-theoretic convention also allows the system matrix in symbolic form to be written down automatically, when use is made of the permutation table of the nodes. In writing down such a matrix, columns corresponding to nodes belonging to the same nodal set must be assigned the same symbol, and we must adopt as many different symbols as there are nodal sets. Thus, if a configuration has six nodal sets, we should have six clusters of columns denoted by the symbols $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{e_i\}$ and $\{f\}$. Each cluster may comprise one or more columns, depending on how many nodes make up the associated nodal set.

Within each cluster of columns having the same symbol, the numerical subscript of the symbol is given by the corresponding number in the permutation table. The only exceptions to this rule relate to instances where a node lies at the centre of symmetry or on an axis of symmetry; in these instances, simple considerations of symmetry must be invoked. Note that the group-theoretic system matrix is not necessarily a symmetric matrix in the conventional sense (a property that stems from the reciprocal theorem), but rather, reflects the physical symmetry of the structural configuration.

For a symmetric system, automatic generation of the group-theoretic system matrix in symbolic form is a central feature of the present scheme. This procedure will become clearer in Section 4, when some illustrative examples are considered.

3. Group-Theoretic transformation matrix and block diagonalization

An outline of the basic concepts of symmetry groups and associated representation theory may be seen in earlier work [13,21,24,31,38], and will not be repeated here. A more detailed coverage of these concepts may be seen in various classical works on the subject [1–4]. Let the character table of a symmetry group G be represented in the form:

$$\begin{array}{c|cccc} & L_1 & L_2 & . \ . & L_k \\ \hline R_1 & L_1^{(1)} & L_2^{(1)} & . \ . & L_k^{(1)} \\ R_2 & L_1^{(2)} & L_2^{(2)} & . \ . & L_k^{(2)} \\ . & . & . & . \ . & . \\ . & . & . & . \ . & . \\ R_k & L_1^{(k)} & L_2^{(k)} & . \ . & L_k^{(k)} \end{array} \tag{1}$$

where $L_1,\ L_2, .., L_k$ denote the k different classes of G, and $R_1,\ R_2, .., R_k$ denote the irreducible representations of G; there are as many classes of G as there are irreducible representations. The orthogonality property of irreducible representations implies that any two rows of the character table are orthogonal. Proof of this may be seen in Ref. [3], among others.

Idempotents [3,10,13,21], which are essentially linear combinations of the symmetry elements of a group, are special operators with the property of generating symmetry-adapted functions when applied to the arbitrary functions ϕ_i ($i = 1, 2, ...N$) of a symmetric problem with N nodes. The idempotents of the centre of the group algebra may be determined from the relationship [3,10,21]:

$$H^{(i)} = \frac{\xi_i}{\xi}\sum_{\sigma} L_i(\sigma^{-1})\sigma \tag{2}$$

where ξ_i is the dimension of the ith irreducible representation (which is given by $L_i(e)$, the first element of the ith row of the character table), ξ is the order of the group G (i.e. the number of elements making up the symmetry group), L_i is the ith character, σ is a symmetry element, and σ^{-1} its inverse. We can write the full set of idempotents of the symmetry group as follows:

$$\begin{Bmatrix} H^{(1)} \\ H^{(2)} \\ . \\ . \\ H^{(k)} \end{Bmatrix} = \begin{bmatrix} \frac{\xi_1}{\xi}L_1^{(1)} & \frac{\xi_1}{\xi}L_2^{(1)} & . \ . & \frac{\xi_1}{\xi}L_k^{(1)} \\ \frac{\xi_2}{\xi}L_1^{(2)} & \frac{\xi_2}{\xi}L_2^{(2)} & . \ . & \frac{\xi_2}{\xi}L_k^{(2)} \\ . & . & . \ . & . \\ . & . & . \ . & . \\ \frac{\xi_k}{\xi}L_1^{(k)} & \frac{\xi_k}{\xi}L_2^{(k)} & . \ . & \frac{\xi_k}{\xi}L_k^{(k)} \end{bmatrix} \begin{Bmatrix} \kappa_1 \\ \kappa_2 \\ . \\ . \\ \kappa_k \end{Bmatrix} \tag{3}$$

that is,

$$\{H\} = [T_H]\{\kappa\} \tag{4}$$

where $\kappa_1,\ \kappa_2, ..,\ \kappa_k$ are class sums (i.e. the sums of the elements of classes $L_1,\ L_2, .., L_k$), and $[T_H]$ is the coefficient matrix of idempotents $\{H\}$.

As an example, consider symmetry group C_{2v} with symmetry elements $\{e,\ C_2,\ \sigma_x,\ \sigma_y\}$. The character table of group C_{2v} is

C_{2v}	e	C_2	σ_x	σ_y
A_1	1	1	1	1
A_2	1	1	−1	−1
B_1	1	−1	1	−1
B_2	1	−1	−1	1

(5)

The group has four irreducible representations $\{R_1,\ R_2,\ R_3,\ R_4\}$, now written in the commonly used convention $\{A_1,\ A_2,\ B_1,\ B_2\}$, and four classes $\{L_1,\ L_2,\ L_3,\ L_4\}$ with one element each:

$$L_1 = \{e\};\ \ L_2 = \{C_2\};\ \ L_3 = \{\sigma_x\};\ \ L_4 = \{\sigma_y\} \tag{6}$$

The group has four idempotents, which we may write as

$$\begin{Bmatrix} H^{(1)} \\ H^{(2)} \\ H^{(3)} \\ H^{(4)} \end{Bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{Bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \\ \kappa_4 \end{Bmatrix} \ i.e.\ \ \{H\} = [T_H]\{\kappa\} \tag{7}$$

where the class sums in this instance consist of only one term each:

$$\kappa_1 = e;\ \ \kappa_2 = C_2;\ \ \kappa_3 = \sigma_x;\ \ \kappa_4 = \sigma_y \tag{8}$$

The coefficient matrix of idempotents is

$$[T_H] = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{9}$$

Returning to the general problem, let the vector space of a physical problem be spanned by independent functions $\phi_1,\ \phi_2,\ ...,\ \phi_n$. These functions may be single freedoms u at each node (linear systems with translations only), or sets of freedoms: $\{u,\ v\}$ (plane systems with translations only); $\{u,\ v,\ w\}$ (space systems with translations only); $\{w,\ \theta\}$ (beam bending); $\{w,\ \theta_x,\ \theta_y\}$ (plate bending); $\{u,\ v,\ w,\ \theta_x,\ \theta_y,\ \theta_x\}$ (general system with translations and rotations). The vector space of the physical problem may therefore be considered to be n-dimensional.

Applying an idempotent of the symmetry group G to each of the n functions of the physical system, we generally obtain n symmetry-adapted functions for the associated subspace, not all of which are independent. From these, we can select r independent symmetry-adapted functions as the basis vectors of the subspace in question. This procedure has been amply illustrated in previous work [13,21,24,26,28,29]. If the problem has k subspaces (as defined by the symmetry group G) denoted by $S^{(1)},\ S^{(2)},\ ...,\ S^{(k)}$, and the dimensions of the subspaces (i.e. the number of symmetry-adapted basis vectors spanning the subspaces) are $r^{(1)},\ r^{(2)},\ ...,\ r^{(k)}$, then $r^{(1)} + r^{(2)} + ... + r^{(k)} = n$. This is the essence of group-theoretic decomposition. We will refer to the symmetry-adapted basis vectors of a subspace as *subspace functions*, to distinguish them from the *conventional functions* of the original space of the problem. Collecting the subspace functions of all subspaces into one column vector, we may write

$$\begin{Bmatrix} \{\Phi^{(1)}\} \\ \{\Phi^{(2)}\} \\ . \\ . \\ \{\Phi^{(k)}\} \end{Bmatrix} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ . \\ . \\ \Phi_n \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & . & . & t_{1,n} \\ t_{2,1} & t_{2,2} & . & . & t_{2,n} \\ . & . & . & . & . \\ . & . & . & . & . \\ t_{n,1} & t_{n,2} & . & . & t_{n,n} \end{bmatrix}\begin{Bmatrix} \phi_1 \\ \phi_2 \\ . \\ . \\ \phi_k \end{Bmatrix} \tag{10}$$

that is,

$$\{\Phi\} = [T]\{\phi\} \tag{11}$$

where $\{\Phi^{(1)}\},\ \{\Phi^{(2)}\},\ ...,\ \{\Phi^{(k)}\}$ are the subspace functions (arranged as a column vector) of subspaces $S^{(1)},\ S^{(2)},\ ...,\ S^{(k)}$ respectively; $\Phi_1,\ \Phi_2,\ ...,\ \Phi_n$ are the same subspace functions now numbered continuously from 1 up to n, starting from the first element of $\{\Phi^{(1)}\}$ up to the last element of $\{\Phi^{(k)}\}$; $\{\phi_1,\ \phi_2,\ ...,\ \phi_n\}$ are the conventional functions of the original problem; $t_{i,j}$ $(i = 1,\ 2,\ ...,\ n;\ j = 1,\ 2,\ ...,\ n)$ are the elements of the square matrix $[T]$ connecting the array $\{\Phi\}$ of subspace functions with the array $\{\phi\}$ of conventional functions.

The matrix $[T]$ has orthogonal properties (this stems directly from the orthogonality of rows of the character table of a symmetry group [3]) and is invertible. In particular, when applied on a stiffness, flexibility, dynamic or stability matrix of a symmetric system formulated in accordance with the group-theoretic convention of Section 2, the matrix $[T]$ converts the system matrix into a non-overlapping block-diagonal form, each block corresponding to an independent subspace of the problem. Thus, $[T]$ is the required group-theoretic transformation matrix for block-diagonalization.

As an example, consider the load–displacement relationship

$$[K]\{\phi\} = \{P\} \tag{12}$$

where $\{\phi\}$ is the column vector of nodal displacements, $\{P\}$ is the column vector of nodal loads, and $[K]$ is the stiffness matrix of the system. The positive directions of the $\{\phi\}$ and $\{P\}$ variables are consistent with the defined group-theoretic convention; $[K]$ is derived on the basis of this convention, and is therefore a *group-theoretic system matrix*. We may rewrite Equation (12) as

$$[K]\left[T^{-1}\right][T]\{\phi\} = \{P\} \tag{13}$$

since the operation $\left[T^{-1}\right][T]$, being equivalent to the identity matrix $[I]$, does not change the value of the left-hand side. Pre-multiplying both sides of Equation (13) by $[T]$, we obtain

$$[T][K]\left[T^{-1}\right][T]\{\phi\} = [T]\{P\} \tag{14}$$

which we may write as

$$[\bar{K}]\{\Phi\} = \{\bar{P}\} \tag{15}$$

where $\{\Phi\}$ is the collection (column vector) of symmetry-adapted freedoms (i.e. subspace functions), and $\{\bar{P}\}$ is the collection (column vector) of symmetry-adapted loads. The symmetry-adapted freedoms and symmetry-adapted loads are the result of the transformation matrix $[T]$ operating on the conventional variables:

$$\{\Phi\} = [T]\{\phi\}\ \ ;\ \ \{\bar{P}\} = [T]\{P\} \tag{16}$$

The matrix $[\bar{K}]$ is a block-diagonal matrix, and a result of the transformation matrix $[T]$ operating on the group-theoretic stiffness matrix:

$$[\bar{K}] = [T][K]\left[T^{-1}\right] \tag{17}$$

It takes the general form

$$[\bar{K}] = \begin{bmatrix} \left[K^{(1)}\right] & 0 & \bullet & \bullet & 0 \\ 0 & \left[K^{(2)}\right] & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \left[K^{(k)}\right] \end{bmatrix} \tag{18}$$

where $\left[K^{(1)}\right],\ \left[K^{(2)}\right],\ ...,\ \left[K^{(k)}\right]$ are the symmetry-adapted stiffness matrices of subspaces $S^{(1)},\ S^{(2)},\ ...,\ S^{(k)}$ respectively; these subspace stiffness matrices are of dimensions $r^{(1)} \times r^{(1)},\ r^{(2)} \times r^{(2)},\ ...,\ r^{(k)} \times r^{(k)}$

respectively. The block-diagonal structure of the $[\bar{K}]$ matrix implies that all elements that are outside the $[K^{(i)}]$ blocks are zeros. The symmetry-adapted system equation in block-diagonal form may finally be written as follows:

$$\begin{bmatrix} [K^{(1)}] & 0 & \bullet & \bullet & 0 \\ 0 & [K^{(2)}] & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & [K^{(k)}] \end{bmatrix} \begin{Bmatrix} \{\Phi^{(1)}\} \\ \{\Phi^{(2)}\} \\ \bullet \\ \bullet \\ \{\Phi^{(k)}\} \end{Bmatrix} = \begin{Bmatrix} \{\mathrm{P}^{(1)}\} \\ \{\mathrm{P}^{(2)}\} \\ \bullet \\ \bullet \\ \{\mathrm{P}^{(k)}\} \end{Bmatrix} \tag{19}$$

It must be noted that the group-theoretic system matrix [K] still retains the property $k_{i,j} = k_{j,i}$ of the conventional stiffness matrix which, of course, stems from the reciprocal theorem or considerations of virtual work. However, the stiffness blocks $K^{(i)}$(symmetric-adapted stiffness matrices for subspaces $S^{(i)}$) may or may not be symmetric, *depending on whether or not the number of components making up each basis vector is the same for all basis vectors of the subspace in question*. This point will be illustrated in the example of Section 4.4.1. It has also been illustrated in previous work [21,38], where symmetry-adapted flexibility matrices $B^{(i)}$ were derived for a square cable net belonging to symmetry group C_{4v}. The fact that $k_{i,j} \neq k_{j,i}$ for some of the subspaces is not at all a disadvantage; what matters is that the size of the problem has been reduced by group-theoretic decomposition through the block-diagonalization of the system stiffness matrix. It is this reduction of the problem into smaller blocks that makes the group-theoretic decomposition computationally advantageous, particularly in the analysis of large-scale problems.

In place of $[K]$, we might have had a flexibility matrix $[A]$, or a determinant for the calculation of eigenvalues (vibration frequencies, buckling loads, etc.). They are all block-diagonalized in the same way, using the group-theoretic transformation matrix $[T]$ for the configuration in question. In the section that follows, we consider some illustrative examples.

4. Illustrative examples

It is believed that the developed formulation is applicable to structural configurations belonging to any of the common symmetry groups (cyclic, dihedral, tetrahedral, octahedral, icosahedral, etc). However, our detailed studies so far have been limited to structural systems with one principal axis of rotational symmetry (typically those belonging to the C_{nv} symmetry groups); the effectiveness of the proposed group-theoretic block-diagonalization procedure for three-dimensional structural systems belonging to higher-order symmetry groups has still to be tested; this is work in progress. We will illustrate the procedure by reference to configurations involving the simple symmetry groups C_{1v} and C_{2v}, for which the results of the decomposition are very easy to visualize.

Also, instead of working with numerical values, we will use algebraic (or symbolic) notation in defining the relevant matrices. The benefit of representing the relevant matrices in symbolic form is that the ensuing results can be regarded as closed-form solutions for the configurations that they represent.

4.1. Example 1: 4-node linear system with C_{1v} symmetry

Let us consider the 4-node horizontal system shown in Fig. 2, which could be a beam with four nodes, a lumped-mass model of a vibrating taut cable, a spring-mass model of a vibrating mechanical system, and so forth. The configuration is symmetrical about the vertical y axis and belongs to symmetry group C_{1v}. This has two symmetry elements $\{e,\ \sigma_y\}$, e being the identity element, and σ_y the reflection in the vertical plane containing the y axis. The centre of symmetry O is taken as the origin of the xy coordinate system, with the positive branch of the x axis being on the right of the y axis. The nodes have been numbered in accordance with the group-theoretic convention defined in Section 2. Table 1 shows how the four nodes are permuted by operations of symmetry group C_{1v}, written in the same order as adopted in the character table of the group.

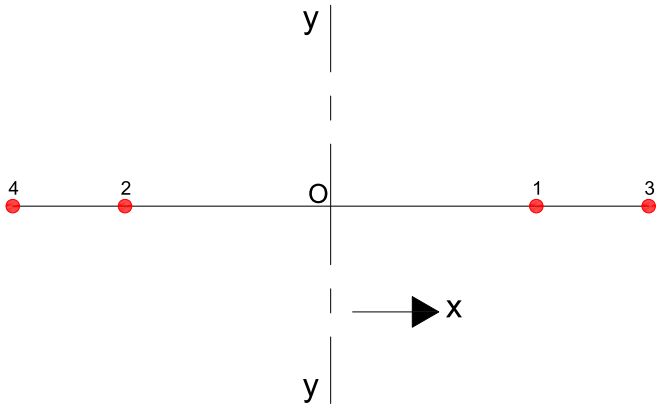

Fig. 2. Example 1: Linear system with 4 nodes.

Table 1
Permutation of nodes of Example 1 under the symmetry operations of group C_{1v}.

node	e	σ_y
1	1	2
2	2	1
3	3	4
4	4	3

As is clear from Fig. 2, the configuration has two nodal sets NS1 and NS2 defined as follows:

$$NS1 = \{1,\ 2\};\ NS2 = \{3,\ 4\} \tag{20}$$

Based purely on conventional considerations of symmetry, and without invoking the reciprocity result (i.e. we do not use the relationship $k_{i,j} = k_{j,i}$, although this still holds), the elements $k_{i,j}$ ($i = 1,\ ..,\ 4;\ j = 1,\ ..,\ 4$) of the conventional stiffness matrix of the system in Fig. 2, assuming one degree of freedom per node, may be written down in terms of eight distinct values or parameters $\{a_1,\ a_2,\ a_3,\ a_4\}$ and $\{b_1,\ b_2,\ b_3,\ b_4\}$ as follows:

$$[K] = \begin{bmatrix} k_{1,1} & k_{1,2} & k_{1,3} & k_{1,4} \\ k_{2,1} & k_{2,2} & k_{2,3} & k_{2,4} \\ k_{3,1} & k_{3,2} & k_{3,3} & k_{3,4} \\ k_{4,1} & k_{4,2} & k_{4,3} & k_{4,4} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & b_1 & b_2 \\ a_2 & a_1 & b_2 & b_1 \\ a_3 & a_4 & b_3 & b_4 \\ a_4 & a_3 & b_4 & b_3 \end{bmatrix} \tag{21}$$

The content of the symbolic stiffness matrix implies that the following relationships holds by virtue of the physical symmetry of the structural configuration:

$$\begin{aligned} k_{1,1} &= k_{2,2} = a_1 \\ k_{1,2} &= k_{2,1} = a_2 \\ k_{3,1} &= k_{4,2} = a_3 \\ k_{3,2} &= k_{4,1} = a_4 \end{aligned} \tag{22a}$$

$$\begin{aligned} k_{1,3} &= k_{2,4} = b_1 \\ k_{1,4} &= k_{2,3} = b_2 \\ k_{3,3} &= k_{4,4} = b_3 \\ k_{3,4} &= k_{4,3} = b_4 \end{aligned} \tag{22b}$$

where the parameters $\{a_1,\ a_2,\ a_3,\ a_4\}$ and $\{b_1,\ b_2,\ b_3,\ b_4\}$ are numerical values of the elements of the conventional stiffness matrix of the

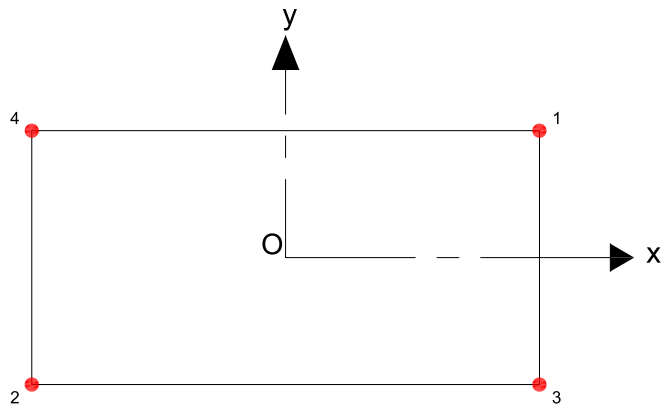

Fig. 3. Example 2: Rectangular system with 4 nodes.

configuration, assumed to be known. This allows us to proceed with the rest of the derivations on the basis of the parameters $\{a_1,\ a_2,\ a_3,\ a_4\}$ and $\{b_1,\ b_2,\ b_3,\ b_4\}$.

In practical calculations, the physical symmetry of the configuration does not actually need to be envisaged in order to write down the symbolic form of the system stiffness matrix. As pointed out at the end of Section 2, the symbolic form of the stiffness matrix of a symmetric system may be written down automatically by making use of the permutation table. By reference to Equation (20), we note that the present example has two nodal sets *NS1* and *NS2*, so we need two sets of algebraic symbols $\{a_i\}$ and $\{b_i\}$. The nodal set *NS1* has the nodes $\{1,\ 2\}$, while the nodal set NS2 has the nodes $\{3,\ 4\}$. Following the procedure indicated at the end of Section 2, we assign the symbol a to Columns 1 and 2 of the system stiffness matrix, and the symbol b to Columns 3 and 4 of the system stiffness matrix. The two columns of the permutation table (Table 1) read $\{1,\ 2,\ 3,\ 4\}^T$ and $\{2,\ 1,\ 4,\ 3\}^T$, so the $\{a\}$ columns will read $\{a_1,\ a_2,\ a_3,\ a_4\}^T$ and $\{a_2,\ a_1,\ a_4,\ a_3\}^T$, while the $\{b\}$ columns will read $\{b_1,\ b_2,\ b_3,\ b_4\}^T$ and $\{b_2,\ b_1,\ b_4,\ b_3\}^T$, which is exactly the result shown in Equation (21).

Thus, by applying the simple rules of Section 2, we may automatically write down the stiffness matrix of a symmetric system in symbolic form without having to think about which coefficients have to be equal from considerations of the physical symmetry of the configuration. In the group-theoretic form of $[K]$, with elements a_i and b_i as shown in Equation (21), the reciprocity relationships $k_{i,j} = k_{j,i}$ hold (as pointed out in Section 3), so that $b_1 = a_3$ and $b_2 = a_4$. However, we do not need to invoke the reciprocity relationships in the present group-theoretic formulation.

The idempotents of group C_{1v} may be written down in terms of the group elements $\{e,\ \sigma_y\}$ as follows:

$$\begin{Bmatrix} H^{(1)} \\ H^{(2)} \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{Bmatrix} e \\ \sigma_y \end{Bmatrix} = [T_H]\{\alpha\} \tag{23}$$

where idempotents $H^{(1)}$ and $H^{(2)}$ correspond to subspaces $S^{(1)}$ and $S^{(2)}$ respectively, and the matrices $[T_H]$ and $\{\alpha\}$ are defined as follows:

$$[T_H] = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad ; \quad \{\alpha\} = \begin{Bmatrix} e \\ \sigma_y \end{Bmatrix} \tag{24}$$

Applying each idempotent to the arbitrary nodal functions $\{\phi_1,\ \phi_2,\ \phi_3,\ \phi_4\}$ of nodes $\{1,\ 2,\ 3,\ 4\}$ respectively, we obtain four symmetry-adapted functions for each subspace, from which two may be selected as the linearly independent basis vectors of the subspace:

$$H^{(1)}\phi_1 = \frac{1}{2}(e+\sigma_y)\phi_1 = \frac{1}{2}(\phi_1+\phi_2) = H^{(1)}\phi_2 \tag{25a}$$

$$H^{(1)}\phi_3 = \frac{1}{2}(e+\sigma_y)\phi_3 = \frac{1}{2}(\phi_3+\phi_4) = H^{(1)}\phi_4 \tag{25b}$$

$$H^{(2)}\phi_1 = \frac{1}{2}(e-\sigma_y)\phi_1 = \frac{1}{2}(\phi_1-\phi_2) = -H^{(2)}\phi_2 \tag{26a}$$

$$H^{(2)}\phi_3 = \frac{1}{2}(e-\sigma_y)\phi_3 = \frac{1}{2}(\phi_3-\phi_4) = -H^{(2)}\phi_4 \tag{26b}$$

$$\begin{Bmatrix} \{\Phi^{(1)}\} \\ \{\Phi^{(2)}\} \end{Bmatrix} = \begin{Bmatrix} \Phi_1^{(1)} \\ \Phi_2^{(1)} \\ \Phi_1^{(2)} \\ \Phi_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}(\phi_1+\phi_2) \\ \frac{1}{2}(\phi_3+\phi_4) \\ \frac{1}{2}(\phi_1-\phi_2) \\ \frac{1}{2}(\phi_3-\phi_4) \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} \tag{27}$$

that is, $\{\Phi\} = [T]\{\phi\}$, where

$$\{\Phi\} = \begin{Bmatrix} \Phi_1^{(1)} \\ \Phi_2^{(1)} \\ \Phi_1^{(2)} \\ \Phi_2^{(2)} \end{Bmatrix} \ ; \ \{\phi\} = \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} \ ; \ [T] = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \tag{28}$$

The inverse of $[T]$ is evaluated to obtain the result:

$$[T^{-1}] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \tag{29}$$

Performing the block-diagonalization transformation on $[K]$, we obtain

$$\begin{aligned} [T][K][T^{-1}] &= \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} a_1 & a_2 & b_1 & b_2 \\ a_2 & a_1 & b_2 & b_1 \\ a_3 & a_4 & b_3 & b_4 \\ a_4 & a_3 & b_4 & b_3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \\ &= \left[\begin{array}{cc:cc} (a_1+a_2) & (b_1+b_2) & & \\ (a_3+a_4) & (b_3+b_4) & & \\ \hdashline & & (a_1-a_2) & (b_1-b_2) \\ & & (a_3-a_4) & (b_3-b_4) \end{array}\right] = [\bar{K}] \end{aligned} \tag{30}$$

Clearly, the $[K]$ matrix has been successfully transformed to the block-diagonalized form $[\bar{K}]$. The symmetry-adapted system equation in block-diagonal form (Equation (19)) becomes.

$$\begin{bmatrix} [K^{(1)}] & [\mathrm{O}] \\ [\mathrm{O}] & [K^{(2)}] \end{bmatrix} \begin{Bmatrix} \{\Phi^{(1)}\} \\ \{\Phi^{(2)}\} \end{Bmatrix} = \begin{Bmatrix} \{P^{(1)}\} \\ \{P^{(2)}\} \end{Bmatrix} \quad \text{or} \quad [\bar{K}]\{\Phi\} = \{\bar{P}\} \tag{31}$$

where

$$[\bar{K}] = \begin{bmatrix} [K^{(1)}] & [\mathrm{O}] \\ [\mathrm{O}] & [K^{(2)}] \end{bmatrix} \; ; \; [K^{(1)}] = \begin{bmatrix} (a_1+a_2) & (b_1+b_2) \\ (a_3+a_4) & (b_3+b_4) \end{bmatrix} \; ; \; [K^{(2)}] = \begin{bmatrix} (a_1-a_2) & (b_1-b_2) \\ (a_3-a_4) & (b_3-b_4) \end{bmatrix} \tag{32}$$

with [O] denoting 2×2 zero matrices, and

$$\{\Phi\} = \begin{Bmatrix} \{\Phi^{(1)}\} \\ \{\Phi^{(2)}\} \end{Bmatrix} = \begin{Bmatrix} \Phi_1^{(1)} \\ \Phi_2^{(1)} \\ \Phi_1^{(2)} \\ \Phi_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}(\phi_1+\phi_2) \\ \frac{1}{2}(\phi_3+\phi_4) \\ \frac{1}{2}(\phi_1-\phi_2) \\ \frac{1}{2}(\phi_3-\phi_4) \end{Bmatrix} \tag{33}$$

$$\{\bar{P}\} = \begin{Bmatrix} \{P^{(1)}\} \\ \{P^{(2)}\} \end{Bmatrix} = \begin{Bmatrix} P_1^{(1)} \\ P_2^{(1)} \\ P_1^{(2)} \\ P_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}(P_1+P_2) \\ \frac{1}{2}(P_3+P_4) \\ \frac{1}{2}(P_1-P_2) \\ \frac{1}{2}(P_3-P_4) \end{Bmatrix} \tag{34}$$

4.2. Example 2: 4-node rectangular system with C_{2v} symmetry

Fig. 3 shows a 4-node rectangular system with C_{2v} symmetry. This could be the joints of a horizontal plane grid subjected to vertical point loads at the joints, the joints of an interior subframe of a multi-storey multi-bay plane frame, the intersections of a 2×2 shallow cable net experiencing small transverse vibrations of four concentrated masses, or the nodes of a finite element. The symmetry group C_{2v} has four elements $\{e,\ C_2,\ \sigma_x,\ \sigma_y\}$ that were described in Section 2.

In accordance with the group-theoretic convention of Section 2, we take the centre of symmetry O of the system as the origin of the xy co-ordinate system, and number the node in the positive-positive quadrant of the xy coordinate system as node 1. The nodes of the system are numbered $\{1,\ 2,\ 3,\ 4\}$ in the order generated by the action of the symmetry elements $\{e,\ C_2,\ \sigma_x,\ \sigma_y\}$ on node 1. Table 2 shows how the four nodes are permuted by the four elements of symmetry group C_{2v}.

The system has only one nodal set:

$$NS1 = \{1,\ 2,\ 3,\ 4\} \tag{35}$$

So, in accordance with the procedure explained in Section 2, only one set of symbols $\{a\}$ is required, and the $[K]$ matrix in symbolic form takes the form

$$[K] = \begin{bmatrix} k_{1,1} & k_{1,2} & k_{1,3} & k_{1,4} \\ k_{2,1} & k_{2,2} & k_{2,3} & k_{2,4} \\ k_{3,1} & k_{3,2} & k_{3,3} & k_{3,4} \\ k_{4,1} & k_{4,2} & k_{4,3} & k_{4,4} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_1 & a_4 & a_3 \\ a_3 & a_4 & a_1 & a_2 \\ a_4 & a_3 & a_2 & a_1 \end{bmatrix} \tag{36}$$

The idempotents $H^{(i)}$ $(i = 1,\ 2,\ 3,\ 4)$ of symmetry group C_{2v} were encountered in Section 2. These may be written directly in terms of the group elements as follows:

$$\begin{Bmatrix} H^{(1)} \\ H^{(2)} \\ H^{(3)} \\ H^{(4)} \end{Bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{Bmatrix} e \\ C_2 \\ \sigma_x \\ \sigma_y \end{Bmatrix} = [T_H]\{\alpha\} \tag{37}$$

where $H^{(1)}$, $H^{(2)}$, $H^{(3)}$ and $H^{(4)}$ correspond to subspaces $S^{(1)}$, $S^{(2)}$, $S^{(3)}$ and $S^{(4)}$, respectively, of the C_{2v} system, and

$$[T_H] = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \; ; \quad \{\alpha\} = \begin{Bmatrix} e \\ C_2 \\ \sigma_x \\ \sigma_y \end{Bmatrix} \tag{38}$$

Applying each idempotent to the nodal functions $\{\phi_1,\ \phi_2,\ \phi_3,\ \phi_4\}$ of nodes $\{1,\ 2,\ 3,\ 4\}$ respectively, we find that there is only one linearly independent basis vector for each subspace (i.e. each subspace is one-dimensional). The independent basis vectors for the four subspaces may be selected, and written down collectively as follows:

$$\begin{Bmatrix} \Phi_1^{(1)} \\ \Phi_1^{(2)} \\ \Phi_1^{(3)} \\ \Phi_1^{(4)} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{4}(\phi_1+\phi_2+\phi_3+\phi_4) \\ \frac{1}{4}(\phi_1+\phi_2-\phi_3-\phi_4) \\ \frac{1}{4}(\phi_1-\phi_2+\phi_3-\phi_4) \\ \frac{1}{4}(\phi_1-\phi_2-\phi_3+\phi_4) \end{Bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} \tag{39}$$

that is, $\{\Phi\} = [T]\{\phi\}$, where

$$\{\Phi\} = \begin{Bmatrix} \Phi_1^{(1)} \\ \Phi_1^{(2)} \\ \Phi_1^{(3)} \\ \Phi_1^{(4)} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{4}(\phi_1+\phi_2+\phi_3+\phi_4) \\ \frac{1}{4}(\phi_1+\phi_2-\phi_3-\phi_4) \\ \frac{1}{4}(\phi_1-\phi_2+\phi_3-\phi_4) \\ \frac{1}{4}(\phi_1-\phi_2-\phi_3+\phi_4) \end{Bmatrix}; \quad \{\phi\} = \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix}; \quad [T] = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{40}$$

Evaluating the inverse of $[T]$, we obtain the result:

$$[T^{-1}] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = 4[T] \tag{41}$$

Performing the block-diagonalization transformation on $[K]$, we obtain

$$[T][K][T^{-1}] = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_1 & a_4 & a_3 \\ a_3 & a_4 & a_1 & a_2 \\ a_4 & a_3 & a_2 & a_1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (a_1+a_2+a_3+a_4) & 0 & 0 & 0 \\ 0 & (a_1+a_2-a_3-a_4) & 0 & 0 \\ 0 & 0 & (a_1-a_2+a_3-a_4) & 0 \\ 0 & 0 & 0 & (a_1-a_2-a_3+a_4) \end{bmatrix} = [\bar{K}] \tag{42}$$

The $[K]$ matrix has been successfully transformed to the block-diagonalized form $[\bar{K}]$. The symmetry-adapted system equation in block-diagonal form (Equation (19)) becomes

$$\begin{bmatrix} K^{(1)} & 0 & 0 & 0 \\ 0 & K^{(2)} & 0 & 0 \\ 0 & 0 & K^{(3)} & 0 \\ 0 & 0 & 0 & K^{(4)} \end{bmatrix}\begin{Bmatrix} \Phi_1^{(1)} \\ \Phi_1^{(2)} \\ \Phi_1^{(3)} \\ \Phi_1^{(4)} \end{Bmatrix} = \begin{Bmatrix} P_1^{(1)} \\ P_1^{(2)} \\ P_1^{(3)} \\ P_1^{(4)} \end{Bmatrix} \quad \text{or} \quad [\bar{K}]\{\Phi\} = \{\bar{P}\} \tag{43}$$

where

$$[\bar{K}] = \begin{bmatrix} K^{(1)} & 0 & 0 & 0 \\ 0 & K^{(2)} & 0 & 0 \\ 0 & 0 & K^{(3)} & 0 \\ 0 & 0 & 0 & K^{(4)} \end{bmatrix} \tag{44}$$

$$K^{(1)} = a_1 + a_2 + a_3 + a_4 \tag{45a}$$

$$K^{(2)} = a_1 + a_2 - a_3 - a_4 \tag{45b}$$

$$K^{(3)} = a_1 - a_2 + a_3 - a_4 \tag{45c}$$

$$K^{(4)} = a_1 - a_2 - a_3 + a_4 \tag{45d}$$

and

Table 2
Permutation of nodes of Example 2 under the symmetry operations of group C_{2v}.

node	e	C_2	σ_x	σ_y
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

$$\{\Phi\} = \begin{Bmatrix} \Phi_1^{(1)} \\ \Phi_1^{(2)} \\ \Phi_1^{(3)} \\ \Phi_1^{(4)} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{4}(\phi_1+\phi_2+\phi_3+\phi_4) \\ \frac{1}{4}(\phi_1+\phi_2-\phi_3-\phi_4) \\ \frac{1}{4}(\phi_1-\phi_2+\phi_3-\phi_4) \\ \frac{1}{4}(\phi_1-\phi_2-\phi_3+\phi_4) \end{Bmatrix} = [T]\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} \tag{46}$$

$$\{\bar{P}\} = \begin{Bmatrix} P_1^{(1)} \\ P_1^{(2)} \\ P_1^{(3)} \\ P_1^{(4)} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{4}(P_1+P_2+P_3+P_4) \\ \frac{1}{4}(P_1+P_2-P_3-P_4) \\ \frac{1}{4}(P_1-P_2+P_3-P_4) \\ \frac{1}{4}(P_1-P_2-P_3+P_4) \end{Bmatrix} = [T]\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} \tag{47}$$

4.3. Example 3: 8-node rectangular system with C_{2v} symmetry

Fig. 4 shows an 8-node rectangular plane grid with C_{2v} symmetry. Defining the centre of symmetry O and the xy coordinate system as shown, the eight nodes have been numbered in accordance with the

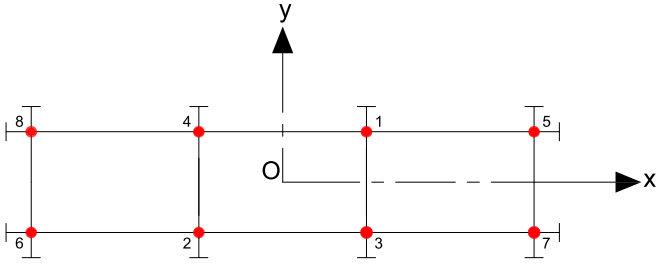

Fig. 4. Example 3: Rectangular plane grid with 8 nodes.

Table 3
Permutation of nodes of Example 3 under the symmetry operations of group C_{2v}.

node	e	C_2	σ_x	σ_y
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1
5	5	6	7	8
6	6	5	8	7
7	7	8	5	6
8	8	7	6	5

group-theoretic convention. Table 3 shows how the eight nodes are permuted by the four elements of symmetry group C_{2v}.

The system has two nodal sets:

$$NS1 = \{1,\ 2,\ 3,\ 4\} \tag{48a}$$

$$NS2 = \{5,\ 6,\ 7,\ 8\} \tag{48b}$$

Therefore the $[K]$ matrix in symbolic form takes the form

$$[K] = \begin{bmatrix} k_{1,1} & k_{1,2} & k_{1,3} & k_{1,4} & k_{1,5} & k_{1,6} & k_{1,7} & k_{1,8} \\ k_{2,1} & k_{2,2} & k_{2,3} & k_{2,4} & k_{2,5} & k_{2,6} & k_{2,7} & k_{2,8} \\ k_{3,1} & k_{3,2} & k_{3,3} & k_{3,4} & k_{3,5} & k_{3,6} & k_{3,7} & k_{3,8} \\ k_{4,1} & k_{4,2} & k_{4,3} & k_{4,4} & k_{4,5} & k_{4,6} & k_{4,7} & k_{4,8} \\ k_{5,1} & k_{5,2} & k_{5,3} & k_{5,4} & k_{5,5} & k_{5,6} & k_{5,7} & k_{5,8} \\ k_{6,1} & k_{6,2} & k_{6,3} & k_{6,4} & k_{6,5} & k_{6,6} & k_{6,7} & k_{6,8} \\ k_{7,1} & k_{7,2} & k_{7,3} & k_{7,4} & k_{7,5} & k_{7,6} & k_{7,7} & k_{7,8} \\ k_{8,1} & k_{8,2} & k_{8,3} & k_{8,4} & k_{8,5} & k_{8,6} & k_{8,7} & k_{8,8} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & b_1 & b_2 & b_3 & b_4 \\ a_2 & a_1 & a_4 & a_3 & b_2 & b_1 & b_4 & b_3 \\ a_3 & a_4 & a_1 & a_2 & b_3 & b_4 & b_1 & b_2 \\ a_4 & a_3 & a_2 & a_1 & b_4 & b_3 & b_2 & b_1 \\ a_5 & a_6 & a_7 & a_8 & b_5 & b_6 & b_7 & b_8 \\ a_6 & a_5 & a_8 & a_7 & b_6 & b_5 & b_8 & b_7 \\ a_7 & a_8 & a_5 & a_6 & b_7 & b_8 & b_5 & b_6 \\ a_8 & a_7 & a_6 & a_5 & b_8 & b_7 & b_6 & b_5 \end{bmatrix} \tag{49}$$

Note that the $[K]$ matrix has two sets of symbols denoted by $\{a\}$ and $\{b\}$, since there are two nodal sets. Nodal set $NS1$ has members $\{1,\ 2,\ 3,\ 4\}$, so Columns 1, 2, 3 and 4 are populated by symbols $\{a\}$; Nodal set $NS2$ has members $\{5,\ 6,\ 7,\ 8\}$, so Columns 5, 6, 7 and 8 are populated by symbols $\{b\}$. The subscript i of the symbols a_i and b_i is given by the number in the corresponding position of the permutation table. This is all consistent with the rules defined earlier, for writing down the system matrix of a symmetric system in symbolic form.

Applying each idempotent of the symmetry group C_{2v} to the nodal functions $\{\phi_1,\ \phi_2,\ \phi_3,\ \phi_4,\ \phi_5,\ \phi_6,\ \phi_7,\ \phi_8\}$ of nodes $\{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8\}$ respectively, we find that there are two linearly independent basis vectors for each subspace (i.e. each subspace is two-dimensional). We may select the two independent basis vectors for each subspace, and collect together all basis vectors of the four subspaces as follows:

$$\begin{Bmatrix} \Phi_1^{(1)} \\ \Phi_2^{(1)} \\ \cdots \\ \Phi_1^{(2)} \\ \Phi_2^{(2)} \\ \cdots \\ \Phi_1^{(3)} \\ \Phi_2^{(3)} \\ \cdots \\ \Phi_1^{(4)} \\ \Phi_2^{(4)} \end{Bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \end{Bmatrix} \tag{50}$$

that is, $\{\Phi\} = [T]\{\phi\}$, where

$$\{\Phi\} = \begin{Bmatrix} \{\Phi^{(1)}\} \\ \{\Phi^{(2)}\} \\ \{\Phi^{(3)}\} \\ \{\Phi^{(4)}\} \end{Bmatrix} = \begin{Bmatrix} \Phi_1^{(1)} \\ \Phi_2^{(1)} \\ \cdots \\ \Phi_1^{(2)} \\ \Phi_2^{(2)} \\ \cdots \\ \Phi_1^{(3)} \\ \Phi_2^{(3)} \\ \cdots \\ \Phi_1^{(4)} \\ \Phi_2^{(4)} \end{Bmatrix}; \quad \{\phi\} = \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \end{Bmatrix}; \quad [T] = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix} \tag{51}$$

Evaluating the inverse of $[T]$, we obtain:

$$[T^{-1}] = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \end{bmatrix} = 4[T^T] \tag{52}$$

Performing the block-diagonalization transformation on $[K]$, we obtain

$$[T][K][T^{-1}] = \left[\begin{array}{cc:cc:cc:cc} A_1^{(1)} & B_1^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ A_2^{(1)} & B_2^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & A_1^{(2)} & B_1^{(2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2^{(2)} & B_2^{(2)} & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & A_1^{(3)} & B_1^{(3)} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_2^{(3)} & B_2^{(3)} & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & A_1^{(4)} & B_1^{(4)} \\ 0 & 0 & 0 & 0 & 0 & 0 & A_2^{(4)} & B_2^{(4)} \end{array}\right] = [\bar{K}] \tag{53}$$

where

$$A_1^{(1)} = a_1 + a_2 + a_3 + a_4 \quad ; \quad B_1^{(1)} = b_1 + b_2 + b_3 + b_4 \tag{54 a, b}$$

$$A_2^{(1)} = a_5 + a_6 + a_7 + a_8 \quad ; \quad B_2^{(1)} = b_5 + b_6 + b_7 + b_8 \tag{54 c, d}$$

$$A_1^{(2)} = a_1 + a_2 - a_3 - a_4 \quad ; \quad B_1^{(2)} = b_1 + b_2 - b_3 - b_4 \tag{55 a, b}$$

$$A_2^{(2)} = a_5 + a_6 - a_7 - a_8 \quad ; \quad B_2^{(2)} = b_5 + b_6 - b_7 - b_8 \tag{55 c, d}$$

$$A_1^{(3)} = a_1 - a_2 + a_3 - a_4 \quad ; \quad B_1^{(3)} = b_1 - b_2 + b_3 - b_4 \tag{56 a, b}$$

$$A_2^{(3)} = a_5 - a_6 + a_7 - a_8 \quad ; \quad B_2^{(3)} = b_5 - b_6 + b_7 - b_8 \tag{56 c, d}$$

$$A_1^{(4)} = a_1 - a_2 - a_3 + a_4 \quad ; \quad B_1^{(4)} = b_1 - b_2 - b_3 + b_4 \tag{57 a, b}$$

$$A_2^{(4)} = a_5 - a_6 - a_7 + a_8 \quad ; \quad B_2^{(4)} = b_5 - b_6 - b_7 + b_8 \tag{57 c, d}$$

The $[K]$ matrix has been successfully transformed to the block-diagonalized form $[\bar{K}]$. The symmetry-adapted system equation in block-diagonal form (Equation (19)) can now be written as

$$\left[\begin{array}{cc:cc:cc:cc} A_1^{(1)} & B_1^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ A_2^{(1)} & B_2^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & A_1^{(2)} & B_1^{(2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2^{(2)} & B_2^{(2)} & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & A_1^{(3)} & B_1^{(3)} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_2^{(3)} & B_2^{(3)} & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & A_1^{(4)} & B_1^{(4)} \\ 0 & 0 & 0 & 0 & 0 & 0 & A_2^{(4)} & B_2^{(4)} \end{array}\right] \left\{\begin{array}{c} \Phi_1^{(1)} \\ \Phi_2^{(1)} \\ \hdashline \Phi_1^{(2)} \\ \Phi_2^{(2)} \\ \hdashline \Phi_1^{(3)} \\ \Phi_2^{(3)} \\ \hdashline \Phi_1^{(4)} \\ \Phi_2^{(4)} \end{array}\right\} = \left\{\begin{array}{c} P_1^{(1)} \\ P_2^{(1)} \\ \hdashline P_1^{(2)} \\ P_2^{(2)} \\ \hdashline P_1^{(3)} \\ P_2^{(3)} \\ \hdashline P_1^{(4)} \\ P_2^{(4)} \end{array}\right\} \tag{58}$$

that is,

$$\begin{bmatrix} [K^{(1)}] & [O] & [O] & [O] \\ [O] & [K^{(2)}] & [O] & [O] \\ [O] & [O] & [K^{(3)}] & [O] \\ [O] & [O] & [O] & [K^{(4)}] \end{bmatrix} \begin{Bmatrix} \{\Phi^{(1)}\} \\ \{\Phi^{(2)}\} \\ \{\Phi^{(3)}\} \\ \{\Phi^{(4)}\} \end{Bmatrix} = \begin{Bmatrix} \{P^{(1)}\} \\ \{P^{(2)}\} \\ \{P^{(3)}\} \\ \{P^{(4)}\} \end{Bmatrix} \tag{59}$$

or simply $[\bar{K}]\{\Phi\} = \{\bar{P}\}$, where

$$[\bar{K}] = \left[\begin{array}{c:c:c:c} [K^{(1)}] & [O] & [O] & [O] \\ \hdashline [O] & [K^{(2)}] & [O] & [O] \\ \hdashline [O] & [O] & [K^{(3)}] & [O] \\ \hdashline [O] & [O] & [O] & [K^{(4)}] \end{array}\right] \tag{60}$$

$$[O] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{61}$$

and the subspace matrices $[K^{(i)}]$ ($i = 1, 2, 3, 4$) are given by

$$\left[K^{(1)}\right]=\begin{bmatrix}A_1^{(1)} & B_1^{(1)}\\ A_2^{(1)} & B_2^{(1)}\end{bmatrix};\quad \left[K^{(2)}\right]=\begin{bmatrix}A_1^{(2)} & B_1^{(2)}\\ A_2^{(2)} & B_2^{(2)}\end{bmatrix};\quad \left[K^{(3)}\right]=\begin{bmatrix}A_1^{(3)} & B_1^{(3)}\\ A_2^{(3)} & B_2^{(3)}\end{bmatrix};\quad \left[K^{(4)}\right]=\begin{bmatrix}A_1^{(4)} & B_1^{(4)}\\ A_2^{(4)} & B_2^{(4)}\end{bmatrix} \tag{62a-d}$$

with the $\left\{A_j^{(i)},\ B_j^{(i)}\right\}$ $(i=1,..,4;\ j=1,\ 2)$ as defined in Equations (54)-(57).

The vectors $\{\Phi\}$ and $\{\bar{P}\}$ are, of course, the symmetry-adapted freedoms and symmetry-adapted loads, respectively, or defined by the following relationships:

$$\left\{\Phi\right\}=\begin{Bmatrix}\{\Phi^{(1)}\}\\ \{\Phi^{(2)}\}\\ \{\Phi^{(3)}\}\\ \{\Phi^{(4)}\}\end{Bmatrix}=\begin{Bmatrix}\Phi_1^{(1)}\\ \Phi_2^{(1)}\\ \Phi_1^{(2)}\\ \Phi_2^{(2)}\\ \Phi_1^{(3)}\\ \Phi_2^{(3)}\\ \Phi_1^{(4)}\\ \Phi_2^{(4)}\end{Bmatrix}=\begin{Bmatrix}\frac{1}{4}\left(\phi_1+\phi_2+\phi_3+\phi_4\right)\\ \frac{1}{4}\left(\phi_5+\phi_6+\phi_7+\phi_8\right)\\ \frac{1}{4}\left(\phi_1+\phi_2-\phi_3-\phi_4\right)\\ \frac{1}{4}\left(\phi_5+\phi_6-\phi_7-\phi_8\right)\\ \frac{1}{4}\left(\phi_1-\phi_2+\phi_3-\phi_4\right)\\ \frac{1}{4}\left(\phi_5-\phi_6+\phi_7-\phi_8\right)\\ \frac{1}{4}\left(\phi_1-\phi_2-\phi_3+\phi_4\right)\\ \frac{1}{4}\left(\phi_5-\phi_6-\phi_7+\phi_8\right)\end{Bmatrix}=\left[T\right]\begin{Bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\\ \phi_6\\ \phi_7\\ \phi_8\end{Bmatrix} \tag{63}$$

$$\left\{\bar{P}\right\}=\begin{Bmatrix}\{P^{(1)}\}\\ \{P^{(2)}\}\\ \{P^{(3)}\}\\ \{P^{(4)}\}\end{Bmatrix}=\begin{Bmatrix}P_1^{(1)}\\ P_2^{(1)}\\ P_1^{(2)}\\ P_2^{(2)}\\ P_1^{(3)}\\ P_2^{(3)}\\ P_1^{(4)}\\ P_2^{(4)}\end{Bmatrix}=\begin{Bmatrix}\frac{1}{4}\left(P_1+P_2+P_3+P_4\right)\\ \frac{1}{4}\left(P_5+P_6+P_7+P_8\right)\\ \frac{1}{4}\left(P_1+P_2-P_3-P_4\right)\\ \frac{1}{4}\left(P_5+P_6-P_7-P_8\right)\\ \frac{1}{4}\left(P_1-P_2+P_3-P_4\right)\\ \frac{1}{4}\left(P_5-P_6+P_7-P_8\right)\\ \frac{1}{4}\left(P_1-P_2-P_3+P_4\right)\\ \frac{1}{4}\left(P_5-P_6-P_7+P_8\right)\end{Bmatrix}=\left[T\right]\begin{Bmatrix}P_1\\ P_2\\ P_3\\ P_4\\ P_5\\ P_6\\ P_7\\ P_8\end{Bmatrix} \tag{64}$$

4.4. Special cases: Nodes lying at the centre of symmetry or in reflection planes

The examples considered so far all have nodes that do not coincide with the centre of symmetry of the configuration, or with planes of reflection symmetry. It should be noted that the entire procedure is also applicable if one or more nodes of the configuration lie at such locations (centre of symmetry, axes of symmetry, planes of reflection). The only (small) difference lies in the way in which *the group-theoretic system matrix in symbolic form* is assembled.

For columns of $[K]$ (the group-theoretic system matrix in symbolic form) which correspond to nodes that do not lie at the centre of symmetry or on axes of symmetry (which are usually the majority of nodes), the algebraic parameters (such as $\{a_i\}$ and $\{b_i\}$) are written down with the aid of the permutation table, as illustrated in the previous examples. However, for columns of $[K]$ which correspond to nodes that lie at the centre of symmetry or on axes of symmetry (which are usually only a few, if any), simple symmetry must be invoked to write down the algebraic parameters. The rest of the steps of the group-theoretic diagonalization procedure remains the same.

In this section, we consider two further examples featuring nodes at the centre of symmetry or on axes of symmetry, and show that the developed group-theoretic procedure successfully achieves block-diagonalization of the structural matrix for such cases as well.

4.4.1. Example 4: Linear system with a node at the centre of symmetry

The C_{1v} linear system in Fig. 5 has five nodes, one of which coincides with the centre of symmetry O, with $y-y$ being the axis of symmetry. The five nodes are numbered in accordance with the convention explained in Section 2, with the result that the central node becomes node 5. Table 4 shows how the five nodes of the configuration are permuted by operations of symmetry group C_{1v}.

The configuration has three nodal sets defined as follows:

$$NS1=\{1,\ 2\};\ NS2=\{3,\ 4\};\ NS3=\{5\} \tag{65}$$

Thus in the 5×5 $[K]$ matrix (the symbolic form of the group-theoretic system matrix), and in accordance with the procedure explained earlier, symbols a_i will be used to denote elements of columns 1 and 2, symbols b_i will denote elements of columns 3 and 4, and symbols c_i will denote elements of column 5. We may therefore write

$$[K]=\begin{bmatrix}k_{1,1} & k_{1,2} & k_{1,3} & k_{1,4} & k_{1,5}\\ k_{2,1} & k_{2,2} & k_{2,3} & k_{2,4} & k_{2,5}\\ k_{3,1} & k_{3,2} & k_{3,3} & k_{3,4} & k_{3,5}\\ k_{4,1} & k_{4,2} & k_{4,3} & k_{4,4} & k_{4,5}\\ k_{5,1} & k_{5,2} & k_{5,3} & k_{5,4} & k_{5,5}\end{bmatrix}=\begin{bmatrix}a_1 & a_2 & b_1 & b_2 & c_1\\ a_2 & a_1 & b_2 & b_1 & c_1\\ a_3 & a_4 & b_3 & b_4 & c_2\\ a_4 & a_3 & b_4 & b_3 & c_2\\ a_5 & a_5 & b_5 & b_5 & c_3\end{bmatrix} \tag{66}$$

with the numerical subscripts of the a_i, b_i and c_i explained as follows: For columns that are associated with nodes that *do not* coincide with the y axis (i.e. columns 1 and 2 representing nodal set *NS1*, and columns 3 and 4 representing nodal set *NS2*), the distinguishing numerical subscripts are written down directly from the permutation table, in accordance with the procedure indicated earlier. However, for columns that are associated with nodes that lie on the centre of symmetry or on axes of symmetry (in the present example, column 5 with the c_i symbols, which represents nodal set *NS3*), the permutation table does not apply (since permutation results are not distinct and repeat under symmetry elements); instead, we make use of very simple symmetry considerations: Imagine a unit action being applied at the central node (i.e. node 5). Clearly, the effect of this action at node 1 (for example, deflection caused) is the same as the effect at node 2 (since nodes 1 and 2 are equidistant from node 5), and similarly the effect of the central action at node 3 is the same as that at node 4 (since nodes 3 and 4 are equidistant from node 5). The effect of the unit action at node 5 (i.e. at the location of the unit action itself) will, of course, have its own distinct value. Therefore, we may write:

$$k_{1,5}=k_{2,5}=c_1 \tag{67a}$$

$$k_{3,5}=k_{4,5}=c_2 \tag{67b}$$

$$k_{5,5}=c_3 \tag{67c}$$

Thus, we have three distinct values of c_i, as shown in the 5th column of the symbolic form of the structural matrix.

As already pointed out, the matrix $[K]$ still retains the property $k_{i,j}=k_{j,i}$ of the conventional stiffness matrix. Thus, in the group-theoretic form of $[K]$, with elements a_i, b_i and c_i as shown in Equation (66), the following relationships certainly hold: $b_1=a_3$, $b_2=a_4$, $c_1=a_5$, and $c_2=b_5$. However, consistent with the present group-theoretic formulation, we do not need to invoke the relationships $k_{i,j}=k_{j,i}$ in writing down the group-theoretic form of $[K]$, though these relationships still hold.

Applying the idempotents of group C_{1v} (Equation (23)) to the nodal functions $\{\phi_1,\ \phi_2,\ \phi_3,\ \phi_4,\ \phi_5\}$ of nodes $\{1,\ 2,\ 3,\ 4,\ 5\}$ respectively, we find that subspace $S^{(1)}$ is 3-dimensional (i.e. it has 3 independent

symmetry-adapted functions), while subspace $S^{(2)}$ is 2-dimensional (only 2 independent symmetry-adapted functions). Basis vectors of the two subspaces may therefore be selected as follows:

$$\left\{\begin{matrix}\{\Phi^{(1)}\}\\ \{\Phi^{(2)}\}\end{matrix}\right\}=\left\{\begin{matrix}\Phi_1^{(1)}\\ \Phi_2^{(1)}\\ \Phi_3^{(1)}\\ \hdashline \Phi_1^{(2)}\\ \Phi_2^{(2)}\end{matrix}\right\}=\left\{\begin{matrix}\frac{1}{2}(\phi_1+\phi_2)\\ \frac{1}{2}(\phi_3+\phi_4)\\ \frac{1}{2}(\phi_5+\phi_5)\\ \hdashline \frac{1}{2}(\phi_1-\phi_2)\\ \frac{1}{2}(\phi_3-\phi_4)\end{matrix}\right\}=\frac{1}{2}\begin{bmatrix}1&1&0&0&0\\0&0&1&1&0\\0&0&0&0&2\\1&-1&0&0&0\\0&0&1&-1&0\end{bmatrix}\left\{\begin{matrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\end{matrix}\right\} \tag{68}$$

that is, $\{\Phi\}=[T]\{\phi\}$, where

$$\{\Phi\}=\left\{\begin{matrix}\Phi_1^{(1)}\\ \Phi_2^{(1)}\\ \Phi_3^{(1)}\\ \hdashline \Phi_1^{(2)}\\ \Phi_2^{(2)}\end{matrix}\right\}=\left\{\begin{matrix}\frac{1}{2}(\phi_1+\phi_2)\\ \frac{1}{2}(\phi_3+\phi_4)\\ \frac{1}{2}(\phi_5+\phi_5)\\ \hdashline \frac{1}{2}(\phi_1-\phi_2)\\ \frac{1}{2}(\phi_3-\phi_4)\end{matrix}\right\}\quad;\quad\{\phi\}=\left\{\begin{matrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\end{matrix}\right\}\quad;\quad[T]=\frac{1}{2}\begin{bmatrix}1&1&0&0&0\\0&0&1&1&0\\0&0&0&0&2\\1&-1&0&0&0\\0&0&1&-1&0\end{bmatrix} \tag{69}$$

Evaluating the inverse of $[T]$, we obtain the result:

$$[T^{-1}]=\begin{bmatrix}1&0&0&1&0\\1&0&0&-1&0\\0&1&0&0&1\\0&1&0&0&-1\\0&0&1&0&0\end{bmatrix} \tag{70}$$

Performing the block-diagonalization transformation on $[K]$, we obtain

$$\begin{aligned}[T][K][T^{-1}] &= \frac{1}{2}\begin{bmatrix}1&1&0&0&0\\0&0&1&1&0\\0&0&0&0&2\\1&-1&0&0&0\\0&0&1&-1&0\end{bmatrix}\begin{bmatrix}a_1&a_2&b_1&b_2&c_1\\a_2&a_1&b_2&b_1&c_1\\a_3&a_4&b_3&b_4&c_2\\a_4&a_3&b_4&b_3&c_2\\a_5&a_5&b_5&b_5&c_3\end{bmatrix}\begin{bmatrix}1&0&0&1&0\\1&0&0&-1&0\\0&1&0&0&1\\0&1&0&0&-1\\0&0&1&0&0\end{bmatrix}\\ &= \left[\begin{array}{ccc:cc}(a_1+a_2)&(b_1+b_2)&c_1&0&0\\(a_3+a_4)&(b_3+b_4)&c_2&0&0\\2a_5&2b_5&c_3&0&0\\ \hdashline 0&0&0&(a_1-a_2)&(b_1-b_2)\\0&0&0&(a_3-a_4)&(b_3-b_4)\end{array}\right]=[\bar{K}]\end{aligned} \tag{71}$$

Thus the $[K]$ matrix has been successfully transformed to the block-diagonalized form $[\bar{K}]$. The symmetry-adapted system equation in block-diagonal form (Equation (19)) can now be written as

$$\left[\begin{array}{ccc:cc}(a_1+a_2)&(b_1+b_2)&c_1&0&0\\(a_3+a_4)&(b_3+b_4)&c_2&0&0\\2a_5&2b_5&c_3&0&0\\ \hdashline 0&0&0&(a_1-a_2)&(b_1-b_2)\\0&0&0&(a_3-a_4)&(b_3-b_4)\end{array}\right]\left\{\begin{matrix}\Phi_1^{(1)}\\ \Phi_2^{(1)}\\ \Phi_3^{(1)}\\ \hdashline \Phi_1^{(2)}\\ \Phi_2^{(2)}\end{matrix}\right\}=\left\{\begin{matrix}P_1^{(1)}\\ P_2^{(1)}\\ P_3^{(1)}\\ \hdashline P_1^{(2)}\\ P_2^{(2)}\end{matrix}\right\} \tag{72}$$

where the stiffness blocks (symmetry-adapted matrices) for subspaces $S^{(1)}$ and $S^{(2)}$ are as follows:

$$K^{(1)}=\begin{bmatrix}(a_1+a_2)&(b_1+b_2)&c_1\\(a_3+a_4)&(b_3+b_4)&c_2\\2a_5&2b_5&c_3\end{bmatrix}; \tag{73a}$$

$$K^{(2)}=\begin{bmatrix}(a_1-a_2)&(b_1-b_2)\\(a_3-a_4)&(b_3-b_4)\end{bmatrix} \tag{73b}$$

and the symmetry-adapted freedoms and symmetry-adapted loads for these subspaces are given by

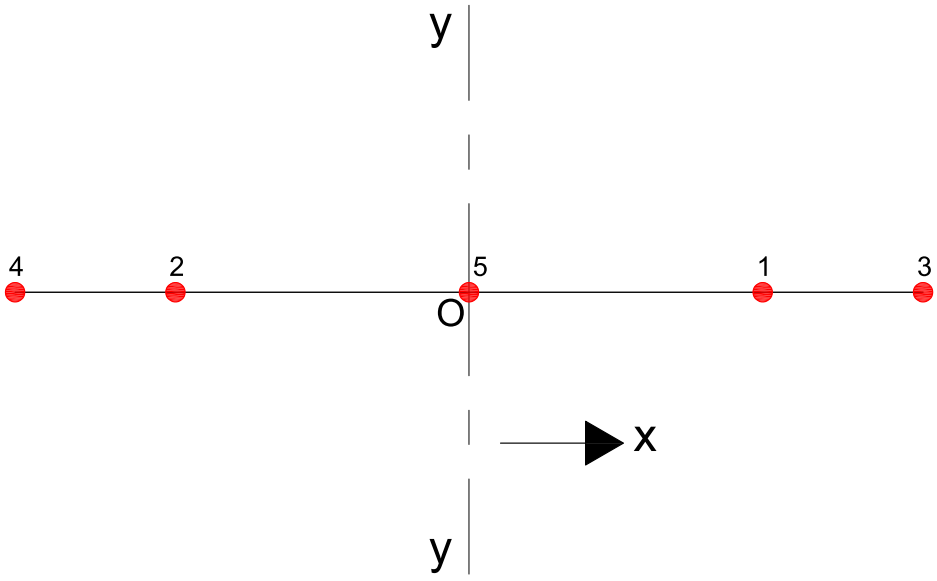

Fig. 5. Example 4: Linear system with a node at the centre.

Table 4
Permutation of nodes of Example 4 under the symmetry operations of group C_{1v}.

node	e	σ_y
1	1	2
2	2	1
3	3	4
4	4	3
5	5	5

$$\{\Phi^{(1)}\} = \begin{Bmatrix} \Phi_1^{(1)} \\ \Phi_2^{(1)} \\ \Phi_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}(\phi_1 + \phi_2) \\ \frac{1}{2}(\phi_3 + \phi_4) \\ \frac{1}{2}(2\phi_5) \end{Bmatrix}; \tag{74a}$$

$$\{\Phi^{(2)}\} = \begin{Bmatrix} \Phi_1^{(2)} \\ \Phi_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}(\phi_1 - \phi_2) \\ \frac{1}{2}(\phi_3 - \phi_4) \end{Bmatrix} \tag{74b}$$

$$\{P^{(1)}\} = \begin{Bmatrix} P_1^{(1)} \\ P_2^{(1)} \\ P_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}(P_1 + P_2) \\ \frac{1}{2}(P_3 + P_4) \\ \frac{1}{2}(2P_5) \end{Bmatrix}; \tag{75a}$$

$$\{P^{(2)}\} = \begin{Bmatrix} P_1^{(2)} \\ P_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}(P_1 - P_2) \\ \frac{1}{2}(P_3 - P_4) \end{Bmatrix} \tag{75b}$$

Note that in equations (73), the stiffness block $K^{(1)}$ (symmetry-adapted stiffness matrix for subspace $S^{(1)}$) is *not* symmetric because the three basis vectors of the subspace $S^{(1)}$ have different numbers of components, while the stiffness block $K^{(2)}$(symmetry-adapted stiffness matrix for subspace $S^{(2)}$) is symmetric because the two basis vectors of the subspace $S^{(2)}$ have the same number of components. For subspace $S^{(1)}$, the basis vector $\Phi_1^{(1)}$ has two components (ϕ_1 and ϕ_2), $\Phi_2^{(1)}$ has two components (ϕ_3 and ϕ_4), but $\Phi_3^{(1)}$ has only one component (ϕ_5) – see Equation (74a). Thus, in the matrix $K^{(1)}$ (see Equation 73a), $k_{1,3} \neq k_{3,1}$ (i.e. $c_1 \neq 2a_5$) and $k_{2,3} \neq k_{3,2}$ (i.e. $c_2 \neq 2b_5$). On the other hand, for subspace $S^{(2)}$, the basis vector $\Phi_1^{(2)}$ has two components (ϕ_1 and ϕ_2), while $\Phi_2^{(2)}$ also has two components (ϕ_3 and ϕ_4) – see Equation (74b). Thus, in the matrix $K^{(2)}$ (see Equation 73b), the relationship $k_{1,2} = k_{2,1}$ (that is, $(b_1 - b_2) = (a_3 - a_4)$) holds. The fact that $k_{i,j} \neq k_{j,i}$ for some of the subspaces is not at all a disadvantage; as pointed out earlier, what matters is that the problem has been decomposed (hence simplified) by group-theoretic block-diagonalization.

4.4.2. *Example 5: Rectangular system with nodes on the axes and at the centre of symmetry*

The C_{2v} rectangular system in Fig. 6 has five nodes: two on the x axis, two on the y axis, and one at the centre of symmetry O. The five nodes are numbered in accordance with the convention explained in Section 2. Table 5 shows how the five nodes of the configuration are permuted by operations of symmetry group C_{2v}.

The configuration has three nodal sets defined as follows:

$$NS1 = \{1,\ 2\};\ NS2 = \{3,\ 4\};\ NS3 = \{5\} \tag{76}$$

Thus, in the 5×5 $[K]$ matrix (the symbolic form of the group-theoretic system matrix), symbols a_i will be used to denote elements of columns 1 and 2, symbols b_i will denote elements of columns 3 and 4, and symbols c_i will denote elements of column 5.

In this example, all nodes either lie on the centre of symmetry or on axes of symmetry. Therefore, we cannot make use of the permutation table to distinguish the various a_i, b_i and c_i (since permutation results are not distinct and repeat under symmetry elements). Instead, we make use of simple symmetry considerations. By reference to Fig. 6, it is only necessary to consider the effects of a unit action at nodes 1, 3 and 5 (i.e. the first nodes of the nodal sets) to generate all the relationships. Considering the effects (e.g. deflections) at nodes 1, 3 and 5 due to unit

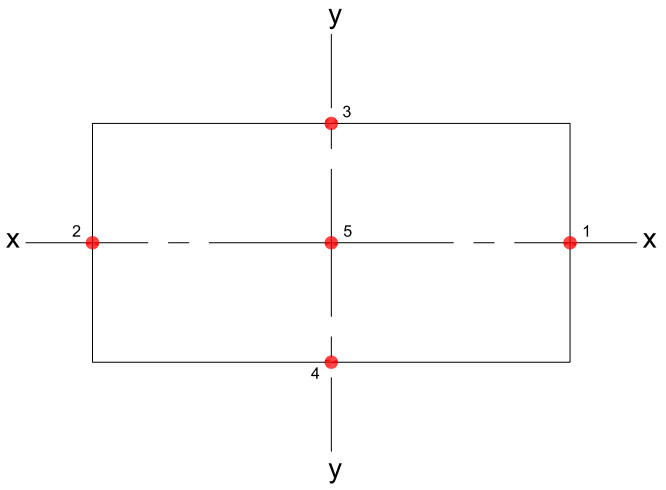

Fig. 6. Example 5: Rectangular system with nodes on the axes and at the centre.

Table 5
Permutation of nodes of Example 5 under the symmetry operations of group C_{2v}.

node	e	C_2	σ_x	σ_y
1	1	2	1	2
2	2	1	2	1
3	3	4	4	3
4	4	3	3	4
5	5	5	5	5

actions placed at nodes 1, 2, 3, 4 and 5 in turn, and using the symmetry of the configuration about the x and y axes, we may write (recalling that symbol a_i must be used for an action at node 1 or node 2, symbol b_i for an action at node 3 or node 4, and symbol c_i for an action at node 5):

Effects at node 1

$$k_{1,1} = k_{2,2} = a_1 \tag{77a}$$

$$k_{1,2} = k_{2,1} = a_2 \tag{77b}$$

$$k_{1,3} = k_{1,4} = k_{2,3} = k_{2,4} = b_1 \tag{77c}$$

$$k_{1,5} = k_{2,5} = c_1 \tag{77d}$$

Effects at node 3

$$k_{3,1} = k_{3,2} = k_{4,1} = k_{4,2} = a_3 \tag{77e}$$

$$k_{3,3} = k_{4,4} = b_2 \tag{77f}$$

$$k_{3,4} = k_{4,3} = b_3 \tag{77g}$$

$$k_{3,5} = k_{4,5} = c_2 \tag{77h}$$

Effects at node 5

$$k_{5,1} = k_{5,2} = a_4 \tag{77i}$$

$$k_{5,3} = k_{5,4} = b_4 \tag{77j}$$

$$k_{5,5} = c_3 \tag{77k}$$

Putting the above relationships together, we have

$$[K] = \begin{bmatrix} k_{1,1} & k_{1,2} & k_{1,3} & k_{1,4} & k_{1,5} \\ k_{2,1} & k_{2,2} & k_{2,3} & k_{2,4} & k_{2,5} \\ k_{3,1} & k_{3,2} & k_{3,3} & k_{3,4} & k_{3,5} \\ k_{4,1} & k_{4,2} & k_{4,3} & k_{4,4} & k_{4,5} \\ k_{5,1} & k_{5,2} & k_{5,3} & k_{5,4} & k_{5,5} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & b_1 & b_1 & c_1 \\ a_2 & a_1 & b_1 & b_1 & c_1 \\ a_3 & a_3 & b_2 & b_3 & c_2 \\ a_3 & a_3 & b_3 & b_2 & c_2 \\ a_4 & a_4 & b_4 & b_4 & c_3 \end{bmatrix} \tag{78}$$

Applying the idempotents of group C_{2v} (Equation (37)) to the nodal functions $\{\phi_1,\ \phi_2,\ \phi_3,\ \phi_4,\ \phi_5\}$ of nodes $\{1,\ 2,\ 3,\ 4,\ 5\}$ respectively, we find that subspace $S^{(1)}$ is 3-dimensional, subspace $S^{(2)}$ is a null subspace (and can therefore be disregarded from this point onwards), subspace $S^{(3)}$ is 1-dimensional, and $S^{(4)}$ is also 1-dimensional. Basis vectors of the three non-zero subspaces may be selected as follows:

$$\begin{Bmatrix} \{\Phi^{(1)}\} \\ \{\Phi^{(3)}\} \\ \{\Phi^{(4)}\} \end{Bmatrix} = \begin{Bmatrix} \Phi_1^{(1)} \\ \Phi_2^{(1)} \\ \Phi_3^{(1)} \\ \Phi_1^{(3)} \\ \Phi_1^{(4)} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}(\phi_1 + \phi_2) \\ \frac{1}{2}(\phi_3 + \phi_4) \\ \frac{1}{2}(\phi_5 + \phi_5) \\ \frac{1}{2}(\phi_1 - \phi_2) \\ \frac{1}{2}(\phi_3 - \phi_4) \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} \tag{79}$$

that is, $\{\Phi\} = [T]\{\phi\}$, where

$$\{\Phi\} = \begin{Bmatrix} \Phi_1^{(1)} \\ \Phi_2^{(1)} \\ \Phi_3^{(1)} \\ \Phi_1^{(3)} \\ \Phi_1^{(4)} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}(\phi_1 + \phi_2) \\ \frac{1}{2}(\phi_3 + \phi_4) \\ \frac{1}{2}(\phi_5 + \phi_5) \\ \frac{1}{2}(\phi_1 - \phi_2) \\ \frac{1}{2}(\phi_3 - \phi_4) \end{Bmatrix} \quad ; \tag{80a}$$

$$\{\phi\} = \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix}, \tag{80b}$$

$$[T] = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \tag{80c}$$

Evaluating the inverse of $[T]$, we obtain the result:

$$[T^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{81}$$

Performing the block-diagonalisation transformation on $[K]$, we obtain.

$$\begin{aligned}\left[T\right]\left[K\right]\left[T^{-1}\right] &= \frac{1}{2}\begin{bmatrix}1&1&0&0&0\\0&0&1&1&0\\0&0&0&0&2\\1&-1&0&0&0\\0&0&1&-1&0\end{bmatrix}\begin{bmatrix}a_1&a_2&b_1&b_1&c_1\\a_2&a_1&b_1&b_1&c_1\\a_3&a_3&b_2&b_3&c_2\\a_3&a_3&b_3&b_2&c_2\\a_4&a_4&b_4&b_4&c_3\end{bmatrix}\begin{bmatrix}1&0&0&1&0\\1&0&0&-1&0\\0&1&0&0&1\\0&1&0&0&-1\\0&0&1&0&0\end{bmatrix} \\ &= \left[\begin{array}{ccc:c:c}(a_1+a_2)&2b_1&c_1&0&0\\2a_3&(b_2+b_3)&c_2&0&0\\2a_4&2b_4&c_3&0&0\\\hdashline 0&0&0&(a_1-a_2)&0\\\hdashline 0&0&0&0&(b_2-b_3)\end{array}\right]=\left[\bar{K}\right]\end{aligned} \tag{82}$$

Thus the $[K]$ matrix has been successfully transformed to the block-diagonalised form $[\bar{K}]$. The symmetry-adapted system equation in block-diagonal form (Equation (19)) can now be written as

$$\left[\begin{array}{ccc:c:c}(a_1+a_2)&2b_1&c_1&0&0\\2a_3&(b_2+b_3)&c_2&0&0\\2a_4&2b_4&c_3&0&0\\\hdashline 0&0&0&(a_1-a_2)&0\\\hdashline 0&0&0&0&(b_2-b_3)\end{array}\right]\left\{\begin{array}{c}\Phi_1^{(1)}\\\Phi_2^{(1)}\\\Phi_3^{(1)}\\\hdashline \Phi_1^{(3)}\\\hdashline \Phi_1^{(4)}\end{array}\right\}=\left\{\begin{array}{c}P_1^{(1)}\\P_2^{(1)}\\P_3^{(1)}\\\hdashline P_1^{(3)}\\\hdashline P_1^{(4)}\end{array}\right\} \tag{83}$$

where the stiffness blocks (symmetry-adapted matrices) for subspaces $S^{(1)}$, $S^{(3)}$ and $S^{(4)}$ are as follows:

$$\begin{aligned}\left[K^{(1)}\right] &= \begin{bmatrix}(a_1+a_2)&2b_1&c_1\\2a_3&(b_2+b_3)&c_2\\2a_4&2b_4&c_3\end{bmatrix} \\ \left[K^{(3)}\right] &= \left[(a_1-a_2)\right] \\ \left[K^{(4)}\right] &= \left[(b_2-b_3)\right]\end{aligned} \tag{84}$$

and the symmetry-adapted freedoms and symmetry-adapted loads for these subspaces are given by

$$\left\{\Phi^{(1)}\right\}=\left\{\begin{array}{c}\Phi_1^{(1)}\\\Phi_2^{(1)}\\\Phi_3^{(1)}\end{array}\right\}=\left\{\begin{array}{c}\frac{1}{2}(\phi_1+\phi_2)\\\frac{1}{2}(\phi_3+\phi_4)\\\frac{1}{2}(2\phi_5)\end{array}\right\} \tag{85a}$$

$$\left\{\Phi^{(3)}\right\}=\left\{\Phi_1^{(3)}\right\}=\left\{\frac{1}{2}(\phi_1-\phi_2)\right\}; \tag{85b}$$

$$\left\{\Phi^{(4)}\right\}=\left\{\Phi_1^{(4)}\right\}=\left\{\frac{1}{2}(\phi_3-\phi_4)\right\} \tag{85c}$$

$$\left\{P^{(1)}\right\}=\left\{\begin{array}{c}P_1^{(1)}\\P_2^{(1)}\\P_3^{(1)}\end{array}\right\}=\left\{\begin{array}{c}\frac{1}{2}(P_1+P_2)\\\frac{1}{2}(P_3+P_4)\\\frac{1}{2}(2P_5)\end{array}\right\} \tag{86a}$$

$$\left\{P^{(3)}\right\}=\left\{P_1^{(3)}\right\}=\left\{\frac{1}{2}(P_1-P_2)\right\}; \tag{86b}$$

$$\left\{P^{(4)}\right\}=\left\{P_1^{(4)}\right\}=\left\{\frac{1}{2}(P_3-P_4)\right\} \tag{86c}$$

Thus, with relatively small adaptations, the developed group-theoretic procedure also successfully achieves block-diagonalization of the structural matrix for those configurations where some of the nodes lie either on axes of symmetry, in reflection planes, or at the centre of symmetry.

5. Validation: Eigenvalues of a rectangular plane grid

To verify the correctness of the outlined block-diagonalization procedure, let us consider the 16-node horizontal rectangular plane grid shown in Fig. 7. For simplicity, and without loss of generality, the applied loads $\{P\}$ will be considered to be vertical point loads (or their equivalent) acting at the grid intersections, while the displacement functions $\{\phi\}$ will be taken as the vertical degrees of freedom at the grid intersections; thus the system has 16 degrees of freedom in total.

If rotational degrees of freedom are ignored, and the system is modelled with three translational degrees of freedom at each node (in the $\{x, y, z\}$ coordinate directions), then the $\{\phi_i\}$ at a given node would simply be the set of freedoms $\{u, v, w\}$ at that node, while the $\{P_i\}$ would be the set of loads $\{P_x, P_y, P_z\}$ at that node. In the most general case, the $\{\phi_i\}$ would comprise three translations and three rotations $\{u_x, u_y, u_z, \theta_x, \theta_y, \theta_z\}$, while the $\{P_i\}$ would comprise three forces and three moments $\{F_x, F_y, F_z, M_x, M_y, M_z\}$. In the present study, and as already pointed out, the simplest model of one freedom per node will be adopted, for the purposes of validating the developed formulation. We will begin by deriving the group-theoretic system matrix via the proposed block-diagonalization procedure, then apply the theoretical formulation to a numerical example, by computing natural frequencies

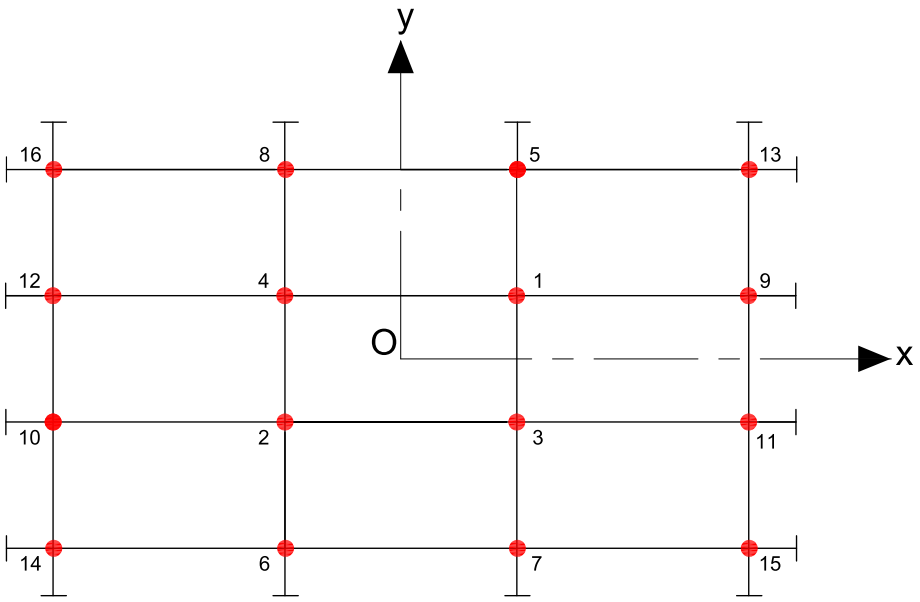

Fig. 7. Rectangular plane grid with 16 nodes: Group-theoretic node numbering.

of vibration for the small transverse motions of the grid. The numerical results of group-theoretic block-diagonalization will be compared with those derived from a conventional analysis.

The configuration clearly has C_{2v} symmetry. Defining the centre of symmetry O and the xy coordinate system as shown, the 16 nodes have been numbered in accordance with the group-theoretic convention. Table 6 shows how the 16 nodes of the grid are permuted by the four symmetry elements of group C_{2v}.

The system has four nodal sets:

$$NS1 = \{1,\ 2,\ 3,\ 4\} \tag{87a}$$

$$NS2 = \{5,\ 6,\ 7,\ 8\} \tag{87b}$$

$$NS3 = \{9,\ 10,\ 11,\ 12\} \tag{87c}$$

$$NS4 = \{13,\ 14,\ 15,\ 16\} \tag{87d}$$

Therefore, in accordance with our group-theoretic convention for column symbols, we may write $[K]$ (the group-theoretic system matrix in symbolic form) as follows

$$[K] = \left[\begin{array}{cccc:cccc:cccc:cccc}
a_1 & a_2 & a_3 & a_4 & b_1 & b_2 & b_3 & b_4 & c_1 & c_2 & c_3 & c_4 & d_1 & d_2 & d_3 & d_4 \\
a_2 & a_1 & a_4 & a_3 & b_2 & b_1 & b_4 & b_3 & c_2 & c_1 & c_4 & c_3 & d_2 & d_1 & d_4 & d_3 \\
a_3 & a_4 & a_1 & a_2 & b_3 & b_4 & b_1 & b_2 & c_3 & c_4 & c_1 & c_2 & d_3 & d_4 & d_1 & d_2 \\
a_4 & a_3 & a_2 & a_1 & b_4 & b_3 & b_2 & b_1 & c_4 & c_3 & c_2 & c_1 & d_4 & d_3 & d_2 & d_1 \\
a_5 & a_6 & a_7 & a_8 & b_5 & b_6 & b_7 & b_8 & c_5 & c_6 & c_7 & c_8 & d_5 & d_6 & d_7 & d_8 \\
a_6 & a_5 & a_8 & a_7 & b_6 & b_5 & b_8 & b_7 & c_6 & c_5 & c_8 & c_7 & d_6 & d_5 & d_8 & d_7 \\
a_7 & a_8 & a_5 & a_6 & b_7 & b_8 & b_5 & b_6 & c_7 & c_8 & c_5 & c_6 & d_7 & d_8 & d_5 & d_6 \\
a_8 & a_7 & a_6 & a_5 & b_8 & b_7 & b_6 & b_5 & c_8 & c_7 & c_6 & c_5 & d_8 & d_7 & d_6 & d_5 \\
a_9 & a_{10} & a_{11} & a_{12} & b_9 & b_{10} & b_{11} & b_{12} & c_9 & c_{10} & c_{11} & c_{12} & d_9 & d_{10} & d_{11} & d_{12} \\
a_{10} & a_9 & a_{12} & a_{11} & b_{10} & b_9 & b_{12} & b_{11} & c_{10} & c_9 & c_{12} & c_{11} & d_{10} & d_9 & d_{12} & d_{11} \\
a_{11} & a_{12} & a_9 & a_{10} & b_{11} & b_{12} & b_9 & b_{10} & c_{11} & c_{12} & c_9 & c_{10} & d_{11} & d_{12} & d_9 & d_{10} \\
a_{12} & a_{11} & a_{10} & a_9 & b_{12} & b_{11} & b_{10} & b_9 & c_{12} & c_{11} & c_{10} & c_9 & d_{12} & d_{11} & d_{10} & d_9 \\
a_{13} & a_{14} & a_{15} & a_{16} & b_{13} & b_{14} & b_{15} & b_{16} & c_{13} & c_{14} & c_{15} & c_{16} & d_{13} & d_{14} & d_{15} & d_{16} \\
a_{14} & a_{13} & a_{16} & a_{15} & b_{14} & b_{13} & b_{16} & b_{15} & c_{14} & c_{13} & c_{16} & c_{15} & d_{14} & d_{13} & d_{16} & d_{15} \\
a_{15} & a_{16} & a_{13} & a_{14} & b_{15} & b_{16} & b_{13} & b_{14} & c_{15} & c_{16} & c_{13} & c_{14} & d_{15} & d_{16} & d_{13} & d_{14} \\
a_{16} & a_{15} & a_{14} & a_{13} & b_{16} & b_{15} & b_{14} & b_{13} & c_{16} & c_{15} & c_{14} & c_{13} & d_{16} & d_{15} & d_{14} & d_{13}
\end{array}\right] \tag{88}$$

Table 6
Permutation of 16 nodes of rectangular plane grid under the symmetry operations of group C_{2v}.

node	e	C_2	σ_x	σ_y
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1
5	5	6	7	8
6	6	5	8	7
7	7	8	5	6
8	8	7	6	5
9	9	10	11	12
10	10	9	12	11
11	11	12	9	10
12	12	11	10	9
13	13	14	15	16
14	14	13	16	15
15	15	16	13	14
16	16	15	14	13

Applying each idempotent of the symmetry group C_{2v} (Equation (37)) to the nodal functions $\{\phi_1, \phi_2, ..., \phi_{16}\}$ of nodes $\{1, 2, ..., 16\}$ respectively, we find that there are four linearly independent symmetry-adapted functions for each subspace $\left\{S^{(1)}, S^{(2)}, S^{(3)}, S^{(4)}\right\}$. Each subspace is thus four-dimensional. We may select four basis vectors for each subspace, and collect together all 16 basis vectors of the four subspaces into one array, as follows:

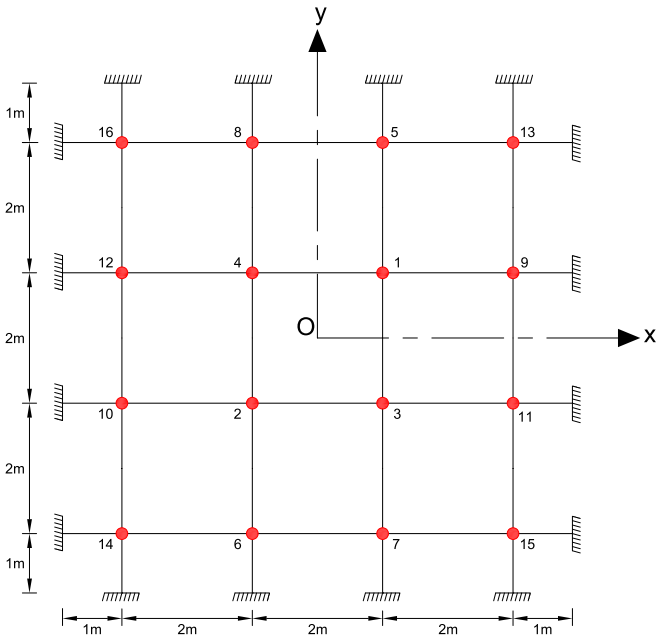

Fig. 8. Square grid with 16 nodes: Numerical example for validation of method.

$$\begin{Bmatrix} \Phi_1^{(1)} \\ \Phi_2^{(1)} \\ \Phi_3^{(1)} \\ \Phi_4^{(1)} \\ \Phi_1^{(2)} \\ \Phi_2^{(2)} \\ \Phi_3^{(2)} \\ \Phi_4^{(2)} \\ \Phi_1^{(3)} \\ \Phi_2^{(3)} \\ \Phi_3^{(3)} \\ \Phi_4^{(3)} \\ \Phi_1^{(4)} \\ \Phi_2^{(4)} \\ \Phi_3^{(4)} \\ \Phi_4^{(4)} \end{Bmatrix} = \frac{1}{4} \left[\begin{array}{cccccccccccccccc} +1 & +1 & +1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & +1 & +1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & +1 & +1 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & +1 & +1 & +1 \\ \hdashline +1 & +1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & +1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & +1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & +1 & -1 & -1 \\ \hdashline +1 & -1 & +1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 & +1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & -1 & +1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & -1 & +1 & -1 \\ \hdashline +1 & -1 & -1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 & -1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & -1 & -1 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & -1 & -1 & +1 \end{array}\right] \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \\ \phi_{10} \\ \phi_{11} \\ \phi_{12} \\ \phi_{13} \\ \phi_{14} \\ \phi_{15} \\ \phi_{16} \end{Bmatrix} \tag{89}$$

that is, $\{\Phi\} = [T]\{\phi\}$, where $[T]$ is the 16×16 square matrix (notice its step-diagonal structure within the subspace compartments separated by the dashed lines). Inverting $[T]$, we get the simple result:

$$\left[T^{-1}\right] = 4\left[T^T\right] \tag{90}$$

Performing the block-diagonalization transformation on $[K]$, we obtain the result

$$[T][K]\left[T^{-1}\right] = \begin{bmatrix} \left[A^{(1)}\right] & [\mathrm{O}] & [\mathrm{O}] & [\mathrm{O}] \\ [\mathrm{O}] & \left[A^{(2)}\right] & [\mathrm{O}] & [\mathrm{O}] \\ [\mathrm{O}] & [\mathrm{O}] & \left[A^{(3)}\right] & [\mathrm{O}] \\ [\mathrm{O}] & [\mathrm{O}] & [\mathrm{O}] & \left[A^{(4)}\right] \end{bmatrix} = \left[\bar{K}\right] \tag{91}$$

where $[\mathrm{O}]$ is a 4×4 zero matrix given by

Table 7
Results of conventional vibration analysis for the 16-node 16 d.o.f. grid experiencing small transverse vibrations.

Mode number	Eigenvalue parameter μ ($\mu = 1000\lambda$)	Frequency parameter ξ ($\xi = q/\sqrt{1000}$)
1	1.74320	0.121
2	0.22110	0.338
3	0.22110	0.338
4	0.11270	0.474
5	0.05346	0.688
6	0.05219	0.697
7	0.04214	0.775
8	0.04214	0.775
9	0.03014	0.917
10	0.03014	0.917
11	0.02760	0.958
12	0.02646	0.978
13	0.02617	0.984
14	0.01926	1.147
15	0.01926	1.147
16	0.01552	1.278

$$[\mathrm{O}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{92}$$

and the $\left[A^{(i)}\right]$ ($i = 1, \ldots, 4$) are 4×4 subspace matrices given by

symmetry of a square grid, but we will use the lower-order symmetry group C_{2v} as we already have the general solution for the block-diagonal matrix of the configuration based on group C_{2v} (i.e. the solution given by Equations (91)-(93)). The eight members making up the grid each have a length of 8.0m. Cross-sectional and material properties of the members are as follows:

Area of cross-section: $\chi = 5 \times 10^{-3}\mathrm{m}^2$
Second moment of area of cross-section: $I = 4.16667 \times 10^{-6}\mathrm{m}^4$
Young's modulus of material: $E = 70 \times 10^{6}\mathrm{kN/m}^2$
Poisson's ratio of material: $\nu = 0.32$

In the dynamic model, we will consider the mass of the members to be lumped at the 16 nodes, each mass having a value of 54kg. The system has 16 degrees of freedom representing the small transverse motions of the 16 masses. This numerical example was considered in a previous study [49].

Let us formulate the free-vibration problem based on the flexibility method rather than the stiffness method. This leads to a system flexibility matrix, which can be block-diagonalized following the procedure that has been developed in the previous sections. The system flexibility matrix, which we will denote by $[F]$, takes the place of $[K]$, and the block-diagonalized flexibility matrix $[\bar{F}]$ is given by

$$\left[\bar{F}\right] = [T][F]\left[T^{-1}\right] = \left[\begin{array}{c|c|c|c} \left[A^{(1)}\right] & [\mathrm{O}] & [\mathrm{O}] & [\mathrm{O}] \\ \hline [\mathrm{O}] & \left[A^{(2)}\right] & [\mathrm{O}] & [\mathrm{O}] \\ \hline [\mathrm{O}] & [\mathrm{O}] & \left[A^{(3)}\right] & [\mathrm{O}] \\ \hline [\mathrm{O}] & [\mathrm{O}] & [\mathrm{O}] & \left[A^{(4)}\right] \end{array}\right] \tag{94}$$

where the $\left[A^{(i)}\right]$ ($i = 1, \ldots, 4$) are now subspace flexibility matrices, with

$$\left[A^{(1)}\right] = \begin{bmatrix} (a_1 + a_2 + a_3 + a_4) & (b_1 + b_2 + b_3 + b_4) & (c_1 + c_2 + c_3 + c_4) & (d_1 + d_2 + d_3 + d_4) \\ (a_5 + a_6 + a_7 + a_8) & (b_5 + b_6 + b_7 + b_8) & (c_5 + c_6 + c_7 + c_8) & (d_5 + d_6 + d_7 + d_8) \\ (a_9 + a_{10} + a_{11} + a_{12}) & (b_9 + b_{10} + b_{11} + b_{12}) & (c_9 + c_{10} + c_{11} + c_{12}) & (d_9 + d_{10} + d_{11} + d_{12}) \\ (a_{13} + a_{14} + a_{15} + a_{16}) & (b_{13} + b_{14} + b_{15} + b_{16}) & (c_{13} + c_{14} + c_{15} + c_{16}) & (d_{13} + d_{14} + d_{15} + d_{16}) \end{bmatrix} \tag{93a}$$

$$\left[A^{(2)}\right] = \begin{bmatrix} (a_1 + a_2 - a_3 - a_4) & (b_1 + b_2 - b_3 - b_4) & (c_1 + c_2 - c_3 - c_4) & (d_1 + d_2 - d_3 - d_4) \\ (a_5 + a_6 - a_7 - a_8) & (b_5 + b_6 - b_7 - b_8) & (c_5 + c_6 - c_7 - c_8) & (d_5 + d_6 - d_7 - d_8) \\ (a_9 + a_{10} - a_{11} - a_{12}) & (b_9 + b_{10} - b_{11} - b_{12}) & (c_9 + c_{10} - c_{11} - c_{12}) & (d_9 + d_{10} - d_{11} - d_{12}) \\ (a_{13} + a_{14} - a_{15} - a_{16}) & (b_{13} + b_{14} - b_{15} - b_{16}) & (c_{13} + c_{14} - c_{15} - c_{16}) & (d_{13} + d_{14} - d_{15} - d_{16}) \end{bmatrix} \tag{93b}$$

$$\left[A^{(3)}\right] = \begin{bmatrix} (a_1 - a_2 + a_3 - a_4) & (b_1 - b_2 + b_3 - b_4) & (c_1 - c_2 + c_3 - c_4) & (d_1 - d_2 + d_3 - d_4) \\ (a_5 - a_6 + a_7 - a_8) & (b_5 - b_6 + b_7 - b_8) & (c_5 - c_6 + c_7 - c_8) & (d_5 - d_6 + d_7 - d_8) \\ (a_9 - a_{10} + a_{11} - a_{12}) & (b_9 - b_{10} + b_{11} - b_{12}) & (c_9 - c_{10} + c_{11} - c_{12}) & (d_9 - d_{10} + d_{11} - d_{12}) \\ (a_{13} - a_{14} + a_{15} - a_{16}) & (b_{13} - b_{14} + b_{15} - b_{16}) & (c_{13} - c_{14} + c_{15} - c_{16}) & (d_{13} - d_{14} + d_{15} - d_{16}) \end{bmatrix} \tag{93c}$$

$$\left[A^{(4)}\right] = \begin{bmatrix} (a_1 - a_2 - a_3 + a_4) & (b_1 - b_2 - b_3 + b_4) & (c_1 - c_2 - c_3 + c_4) & (d_1 - d_2 - d_3 + d_4) \\ (a_5 - a_6 - a_7 + a_8) & (b_5 - b_6 - b_7 + b_8) & (c_5 - c_6 - c_7 + c_8) & (d_5 - d_6 - d_7 + d_8) \\ (a_9 - a_{10} - a_{11} + a_{12}) & (b_9 - b_{10} - b_{11} + b_{12}) & (c_9 - c_{10} - c_{11} + c_{12}) & (d_9 - d_{10} - d_{11} + d_{12}) \\ (a_{13} - a_{14} - a_{15} + a_{16}) & (b_{13} - b_{14} - b_{15} + b_{16}) & (c_{13} - c_{14} - c_{15} + c_{16}) & (d_{13} - d_{14} - d_{15} + d_{16}) \end{bmatrix} \tag{93d}$$

Let us now apply the developed expressions to the small vertical vibrations of the plane grid shown in Fig. 8, where the members are spaced at 2m in both the x and y directions. The members are simply supported at their ends, and rigidly intersect at 16 nodes numbered in accordance with the group-theoretic convention when use is made of symmetry group C_{2v}. Note that this configuration actually has the C_{4v} elements that are linear combinations of system flexibility coefficients exactly in the pattern given by Equations (93).

Based purely on conventional considerations, if we apply a unit vertical force at Node j of the grid ($j = 1, 2, \ldots, 16$), and note the ensuing vertical deflection at Node i of the grid ($i = 1, 2, \ldots, 16$), we can assemble all 256 f_{ij} elements of the conventional system flexibility matrix $[F]$. The system matrix has the same symbolic form regardless of

whether the system matrix is a stiffness matrix or a flexibility matrix. Thus, the group-theoretic system flexibility matrix $[F]$ also has the form given by Equation (88).

By applying a unit vertical force of 1.0N at representative nodes of the grid, we may easily calculate the 64 distinct parameters $\{a_r,\ b_r,\ c_r,\ d_r\}$ (where $r = 1,\ 2,\ \ldots,\ 16$) in the matrix of Equation (88), noting that each of these 64 parameters repeats four times in the system flexibility matrix $[F]$ to generate all the 256 flexibility coefficients f_{ij} ($i = 1,\ 2,\ \ldots,\ 16;\ j = 1,\ 2,\ \ldots,\ 16$) of the system. The results (in units of 10^{-6}m/N) are as follows:

$$\begin{array}{llll} a_1 = 7.13 & a_2 = 5.09 & a_3 = 5.70 & a_4 = 5.70 \\ a_5 = 3.03 & a_6 = 1.95 & a_7 = 2.05 & a_8 = 2.49 \\ a_9 = 3.03 & a_{10} = 1.95 & a_{11} = 2.49 & a_{12} = 2.05 \\ a_{13} = 1.35 & a_{14} = 0.78 & a_{15} = 0.89 & a_{16} = 0.89 \\ b_1 = 3.03 & b_2 = 1.95 & b_3 = 2.05 & b_4 = 2.49 \\ b_5 = 2.16 & b_6 = 0.71 & b_7 = 0.71 & b_8 = 1.32 \\ b_9 = 1.35 & b_{10} = 0.78 & b_{11} = 0.89 & b_{12} = 0.89 \\ b_{13} = 0.84 & b_{14} = 0.29 & b_{15} = 0.29 & b_{16} = 0.38 \end{array}$$

$$\begin{array}{llll} c_1 = 3.03 & c_2 = 1.95 & c_3 = 2.49 & c_4 = 2.05 \\ c_5 = 1.35 & c_6 = 0.78 & c_7 = 0.89 & c_8 = 0.89 \\ c_9 = 2.16 & c_{10} = 0.71 & c_{11} = 1.32 & c_{12} = 0.71 \\ c_{13} = 0.84 & c_{14} = 0.29 & c_{15} = 0.38 & c_{16} = 0.29 \\ d_1 = 1.35 & d_2 = 0.78 & d_3 = 0.89 & d_4 = 0.89 \\ d_5 = 0.84 & d_6 = 0.29 & d_7 = 0.29 & d_8 = 0.38 \\ d_9 = 0.84 & d_{10} = 0.29 & d_{11} = 0.38 & d_{12} = 0.29 \\ d_{13} = 0.97 & d_{14} = 0.12 & d_{15} = 0.12 & d_{16} = 0.12 \end{array}$$

Next, we evaluate the elements of the subspace flexibility matrices $\left[A^{(i)}\right]$ ($i = 1,\ \ldots,\ 4$) from the linear combinations of Equations (93), and using the above numerical values:

$$\left[A^{(1)}\right] = \begin{bmatrix} 23.62 & 9.52 & 9.52 & 3.91 \\ 9.52 & 4.90 & 3.91 & 1.80 \\ 9.52 & 3.91 & 4.90 & 1.80 \\ 3.91 & 1.80 & 1.80 & 1.33 \end{bmatrix};$$

$$\left[A^{(2)}\right] = \begin{bmatrix} 0.82 & 0.44 & 0.44 & 0.35 \\ 0.44 & 0.84 & 0.35 & 0.46 \\ 0.44 & 0.35 & 0.84 & 0.46 \\ 0.35 & 0.46 & 0.46 & 0.85 \end{bmatrix}$$

$$\left[A^{(3)}\right] = \begin{bmatrix} 2.04 & 0.64 & 1.52 & 0.57 \\ 0.64 & 0.84 & 0.57 & 0.46 \\ 1.52 & 0.57 & 2.06 & 0.64 \\ 0.57 & 0.46 & 0.64 & 0.85 \end{bmatrix};$$

$$\left[A^{(4)}\right] = \begin{bmatrix} 2.04 & 1.52 & 0.64 & 0.57 \\ 1.52 & 2.06 & 0.57 & 0.64 \\ 0.64 & 0.57 & 0.84 & 0.46 \\ 0.57 & 0.64 & 0.46 & 0.85 \end{bmatrix}$$

For each subspace $S^{(i)}$ ($i = 1,\ \ldots,\ 4$), the mass matrix $M^{(i)}$ is a diagonal matrix whose non-zero diagonal elements are the values of the nodal masses at each nodal set associated with the subspace. In the present example, all four subspaces are associated with the same nodal sets as defined by Equations (87), and the value of the nodal mass is the same for all four nodal sets, and equal to 54kg. Thus,

$$M^{(i)} = \begin{bmatrix} 54 & 0 & 0 & 0 \\ 0 & 54 & 0 & 0 \\ 0 & 0 & 54 & 0 \\ 0 & 0 & 0 & 54 \end{bmatrix} \quad ; \quad (i = 1,\ 2,\ 3,\ 4)$$

The 16 eigenvalues for the free vibration response of the entire system are generated by solving the 4-dimensional eigenvalue problem for each subspace (independently of other subspaces), and putting all subspace solutions together to give the full set of eigenvalues for the system; this procedure has been explained in previous work [21,30,38,49]. Taking one subspace at a time, the eigenvalues associated with the subspace are obtained from the vanishing condition

$$\left|\left[A^{(i)}\right] - \lambda\left[M^{(i)}\right]^{-1}\right| = 0 \tag{95}$$

Substituting the numerical values of $\left[A^{(i)}\right]$ and $M^{(i)}$ ($i = 1,\ \ldots,\ 4$) into the above equation, and expanding the determinant, we obtain a 4th-degree characteristic equation for each subspace, which upon solving yields four roots for λ, hence four natural frequencies of vibration q $\left(= 1/2\pi\sqrt{\lambda}\right)$ as follows (with q being in Hz):

Subspaces $S^{(1)}$.

$\lambda_1 = 1.7432 \times 10^{-3}$ $q_1 = 0.121\sqrt{1000}$ $\lambda_2 = 0.02760 \times 10^{-3}$ $q_2 = 0.958\sqrt{1000}$ $\lambda_3 = 0.05219 \times 10^{-3}$ $q_3 = 0.697\sqrt{1000}$ $\lambda_4 = 0.05346 \times 10^{-3}$ $q_4 = 0.688\sqrt{1000}$.

Subspaces $S^{(2)}$.

$\lambda_1 = 0.1127 \times 10^{-3}$ $q_1 = 0.474\sqrt{1000}$ $\lambda_2 = 0.01552 \times 10^{-3}$ $q_2 = 1.278\sqrt{1000}$ $\lambda_3 = 0.02617 \times 10^{-3}$ $q_3 = 0.984\sqrt{1000}$ $\lambda_4 = 0.02646 \times 10^{-3}$ $q_4 = 0.978\sqrt{1000}$.

Subspaces $S^{(3)}$.

$\lambda_1 = 0.2211 \times 10^{-3}$ $q_1 = 0.338\sqrt{1000}$ $\lambda_2 = 0.04214 \times 10^{-3}$ $q_2 = 0.775\sqrt{1000}$ $\lambda_3 = 0.01926 \times 10^{-3}$ $q_3 = 1.147\sqrt{1000}$ $\lambda_4 = 0.03014 \times 10^{-3}$ $q_4 = 0.917\sqrt{1000}$.

Subspaces $S^{(4)}$.

$\lambda_1 = 0.2211 \times 10^{-3}$ $q_1 = 0.338\sqrt{1000}$ $\lambda_2 = 0.04214 \times 10^{-3}$ $q_2 = 0.775\sqrt{1000}$ $\lambda_3 = 0.01926 \times 10^{-3}$ $q_3 = 1.147\sqrt{1000}$ $\lambda_4 = 0.03014 \times 10^{-3}$ $q_4 = 0.917\sqrt{1000}$.

Conventional vibration analysis, without taking advantage of the symmetry of the configuration, yields the eigenvalues shown in Table 7, with the natural frequencies being given in ascending order:

The natural frequencies calculated via the developed group-theoretic block-diagonalization of the system flexibility matrix are identical to those obtained from a conventional vibration analysis, and are also similar to those obtained using a group-theoretic procedure that does not involve block-diagonalization [49]. This shows that the developed procedure can be relied upon to yield the correct results.

6. Concluding remarks

For structural problems exhibiting symmetry properties, group theory allows computational simplifications to be achieved by decomposing the vector space of the problem into smaller subspaces that are independent of each other. When the basis vectors of a subspace are used as the symmetry-adapted variables of that subspace, a smaller problem (associated with a matrix of smaller dimensions) automatically results. The same decomposition may be achieved by first obtaining the structural matrix of the system, and then transforming this into a non-overlapping block-diagonal matrix, each independent block being associated with a subspace of the problem.

The first approach has the advantage of being computationally more efficient. It is not necessary to assemble the structural matrix of the full system first; subspace matrices are directly computed by noting the effects of symmetry-adapted functions (i.e. within the subspaces, symmetry-adapted functions take the place of conventional variables). The disadvantage is that implementation requires some degree of physical visualization (basis vectors are usually plotted to assist this process); computer programs are not well-suited to this. On the other hand, block-diagonalization is computationally more demanding, since

the conventional structural matrix has to be first assembled, then converted into block-diagonal form through a suitable transformation operation. The advantage is that once the transformation matrix has been correctly set up, block-diagonalization is achieved through a series of standard matric operations, with not much physical imagination required. This makes the second approach more amenable to computer programming. Currently, no simple and consistent group-theoretic formulation exists for achieving this block-diagonalization.

This contribution has presented a robust and consistent group-theoretic block-diagonalization of structural matrices. The formulation is different from existing block-diagonalization procedures in two important respects: (i) a very specific convention for choosing the origin of the global coordinate system, numbering the nodes of the structural system, and choosing the positive directions of the freedoms and loads at the nodes, is required; (ii) the transformation matrix is assembled as a square matrix connecting the array of all subspace basis vectors (arranged in a very specific manner) with the array of conventional functions of the full vector space of the system.

Some simple examples involving symmetry groups groups C_{1v} and C_{2v} have been considered to illustrate the computation of block-diagonal structural matrices using the developed procedure. The formulation has been validated through consideration of a numerical example of a rectangular plane grid experiencing small transverse motions; natural frequencies calculated via the developed group-theoretic block-diagonalization procedure have been found to be identical to those obtained from a conventional analysis that does not take advantage of the symmetry of the structural problem, showing that the developed procedure is reliable.

For configurations having more than one axis of rotational symmetry (such as those belonging to symmetry groups D_{nh}, T_h, O_h and I_h), more care is needed, but the process of node numbering remains the same. We begin by defining the centre of symmetry O, and then label all rotation axes and reflection planes of the configuration in accordance with the notation of the character table of the symmetry group. To generate a consistent group-theoretic numbering of all the nodes of the system, we simply adopt the order of operations as given across the top of the character table of that group. As an example, consider a configuration that conforms to the symmetry of a rectangular prism. Such a configuration has three axes of rotational symmetry (i.e. x, y, z axes), each associated with a C_2 operation (i.e. a rotation of π). This configuration belongs to the symmetry group D_{2h} of order 8. For this group, the 8 elements appear across the top of the character table in the order $\{E, C_2^z, C_2^y, C_2^x, i, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}\}$, so in the group-theoretic node-numbering process, symmetry operations on initial nodes (in the positive-positive-positive octant) should be executed exactly in that order: identity operation (initial position unmoved); π rotation about the z axis; π rotation about the y axis; π rotation about the x axis; inversion through the centre of symmetry O; reflection in the plane containing O and the $\{x, y\}$ axes; reflection in the plane containing O and the $\{x, z\}$ axes; reflection in the plane containing O and the $\{y, z\}$. Idempotents of the symmetry group are then applied to each of the n functions of the physical system to obtain n symmetry-adapted functions for the associated subspace, from which r independent symmetry-adapted functions are selected as the basis vectors of the subspace. Collecting all basis vectors into an array then yields the transformation matrix T in the manner already described. So the basic approach remains the same for configurations having more than one axis of rotational symmetry, but care is needed to ensure that the order of execution of symmetry operations (including the order in which the rotation axes are considered) remains the same throughout the whole process.

The whole group-theoretic computational process (node numbering, derivation of the transformation matrix, transformation of the conventional structural matrix to block-diagonal form, and execution of the decomposed problem within the independent subspaces) can be implemented automatically through computer programming. This development is still work in progress. It is hoped that this contribution is a significant step towards the implementation of group-theoretic formulations in computational schemes for practical engineering analyses.

CRediT authorship contribution statement

Alphose Zingoni: Conceptualization, Methodology, Investigation, Formal analysis, Writing – original draft, Writing – review & editing. **Chisanga Kaluba:** Validation.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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